Optimal monetary-fiscal policy with zero lower bound and fiscal limit constraints

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Abstract

With their aging populations, many advanced economies are currently facing (i) a decline in their long-run interest rates, and (ii) a strong increase in their social security deficit. The former increases the probability that the policy rate is constrained by the zero lower bound (ZLB). The latter increases the probability that the fiscal limit (FL) is reached. In this paper, we study the optimal monetary-fiscal policy under both constraints. We conduct our analysis in an extension of the basic New Keynesian model with an endogenous FL. We assume away both outright default on public debt and outright monetary financing. Two of the main insights are: (i) In response to negative demand shocks, as the economy approaches FL, the reduction in fiscal space limits the future "output boom" that the policymaker can promise at the ZLB. Subsequently, real activity is less stabilized in a liquidity trap. (ii) Positive technology shocks temporarily shift the Laffer curve upwards, and, relatively to demand shocks, allow the policymaker to attain a better welfare outcome in the proximity of the FL when the ZLB is binding.[1]

Keywords: Monetary policy, fiscal policy, endogenous fiscal limit, ZLB

JEL Class: E2 – E3 – E4.

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1 Introduction

Many advanced economies are currently (i) facing a decline in long-run interest rates, and (ii) approaching their fiscal limit. One common driving force of these structural long-run trends is population aging. Specifically, the *secular stagnation* literature (e.g. Summers (2014), Eggertsson and Mehrotra (2014), Gordon (2016)) emphasizes that these demographic changes, alongside other factors, push long-run interest rates downwards (figure 1). The argument goes that the decline in the ratio of young to middle–aged reduces loan demand, and hence, triggers a decline in long-run interest rates (Eggertsson, Mehrotra and Robbins (2019)). At the same time, the *fiscal limit* literature warns that the rise in old-age dependency ratios in advanced economies (figure 2) requires high tax adjustments to finance adequate pensions, health and long-term care expenditures (figure 3), and thus pushes government finances on unsustainable paths (e.g. Davig, Leeper and Walker (2010), Leeper and Walker (2011)).

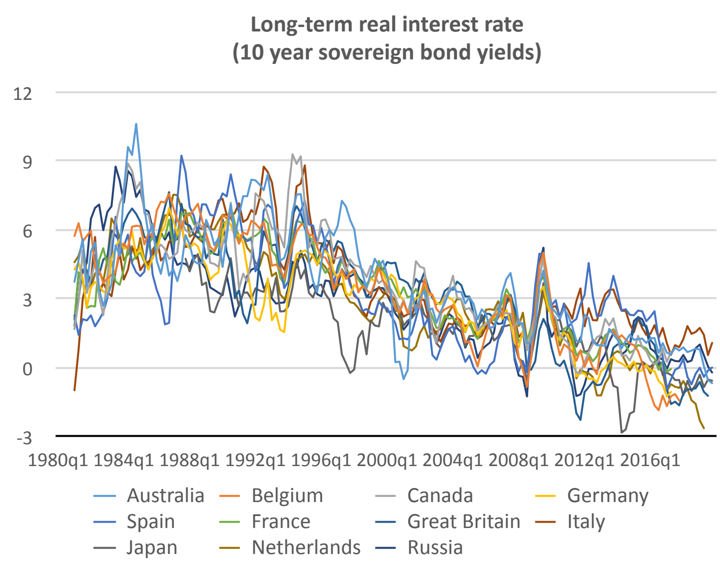

Figure 1: Real interest rates in advanced economies
Source: Bank for International Settlements

Against this background, in the present paper we take the two ongoing structural trends as given, and ask how they *jointly* affect the optimal policy response to business cycle fluctuations. Literature so far has focused exclusively on how lower long-run interest rates affected optimal policy, and, more specifically, on how optimal policy looked like in a world where the ZLB bound more often. But it did not study how the character of optimal policy at the ZLB might change as the economy converged to its long-run fiscal limit. Answering this question is thus our contribution to the literature. We address this issue through the lenses of the standard analytical framework in the literature, the New-Keynesian model, which we extend with an endogenous long-run fiscal limit. To our knowledge, endogenous fiscal limits have been used so far only in real models (e.g. Schmitt-Grohé (1997) or Bi (2012)). We are the first to consider it explicitly in a monetary model to study how monetary policy can alter its shape, making our analytical setup on its own an innovation with respect to existing literature. We consider two types of real disturbances: a negative demand preference shock, and a positive technology shock, both of which are strong enough to drive the economy in a liquidity trap. We assume away both outright default on public debt and outright monetary financing[2].

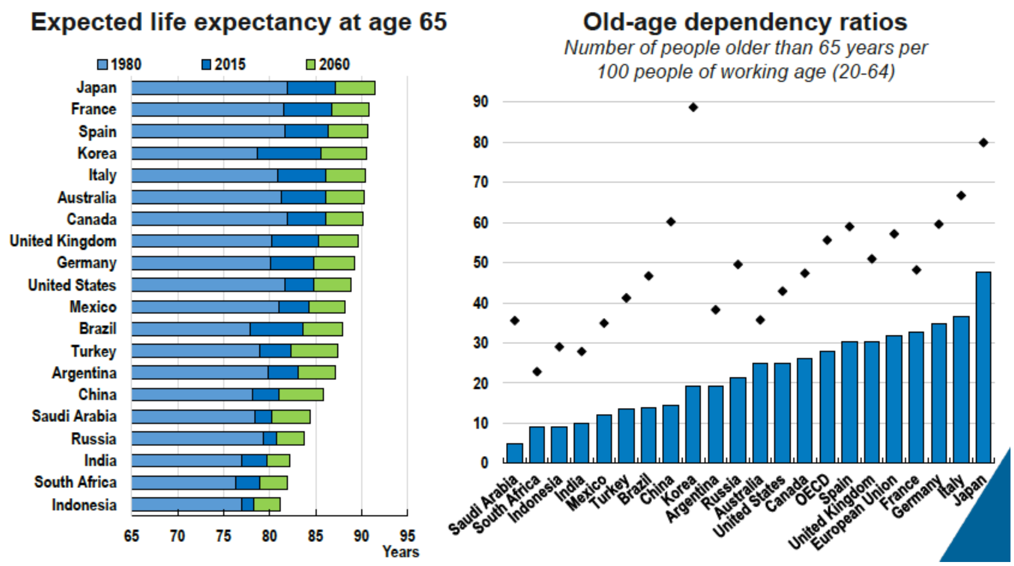

Figure 2: Aging populations in OECD countries (◇ projected values for 2060)
Source: OECD (2019) – Fiscal changes and inclusive growth in aging societies

At first sight, the relevance for optimal policy of the decline in long-run interest rates may be more obvious than the one of the convergence to the fiscal limit. Specifically, lower long-run interest rates imply that for a given distribution of shocks, the policy rate has to decline on average more so as to track the efficient interest rate and approach the economy to its efficient allocation. But since

[2]Since in our analysis we consider a consolidated balance-sheet between the government and the central bank, and public debt is nominal, the equilibrium inflation resulted from monetary policy decisions acts *de facto* as monetary financing in our setup. Thus, while our model is “cash-less”, and hence abstracts away from the use ”seignorage revenue” (resulted from the variation of central-bank banknotes) to finance public debt (what is generally know as “monetary financing”), the policymaker still has access in our framework to a more subtle way to “monetary finance” the public debt via inflation in the spirit of the fiscal theory of the price level (e.g. Leeper (1991), Sims (1994)).

the policy rate cannot decrease below zero (in principle), this implies that it will hit the ZLB more often, creating a challenge for policy. In this context, literature has studied how the ZLB affected the optimal business-cycle policy. In a companion paper where we abstract from the ZLB, we show however that the convergence to the fiscal limit has also implications for the optimal response to business cycle fluctuations. Specifically, tax variations affect differently macro-variables, as well as marginal tax revenues, depending on the proximity of the economy to the fiscal limit. For instance, the closer the economy to the fiscal limit, the smaller the marginal effects on tax revenues, and the stronger the impact on price inflation of a tax hike[3]. These structural changes, together with all others related to the convergence path to the fiscal limit, generally affect the nature of optimal policy and its welfare outcome.

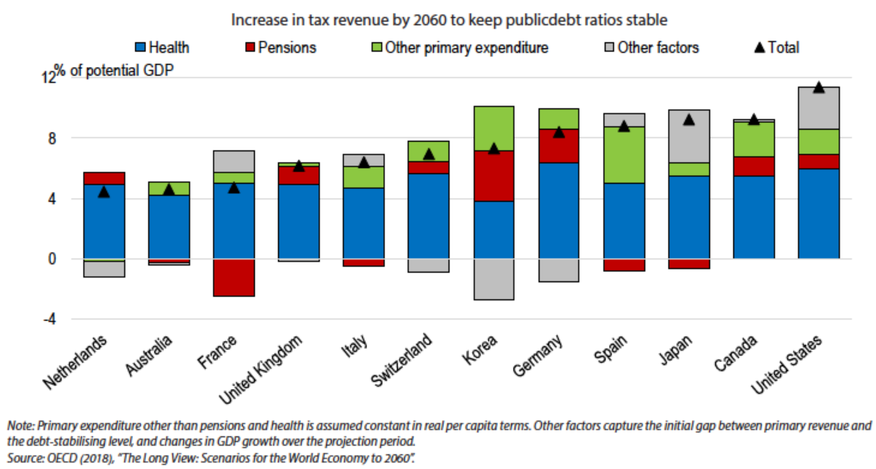

Figure 3: Steep tax increases needed to absorb ageing costs
Source: OECD (2019) – Fiscal changes and inclusive growth in aging societies

Our analysis outlines so far two main insights on how the convergence to the fiscal limit affects the optimal business cycle policy at the ZLB (i.e. in a "liquidity trap")[4]. One main insight concerns negative demand preference shocks. Specifically, in response to such shocks, as the economy approaches fiscal limit, the reduction in fiscal space limits the future output boom that the policymaker can promise at the ZLB. Thus, under optimal policy, underutilization of production capacity can be less mitigated during the liquidity trap, and welfare losses increase significantly. The explanation goes as follows. As already pointed out by Eggertsson and Woodford (2003), taxes

[3]The effect on price sensitivity applies to taxes affecting the supply-side of the economy.

[4]Throughout the analysis we assume that the joint monetary-fiscal authority can commit, and we study, following previous literature, optimal policy from a "timeless perspective" (Woodford (1999; 2003, chapter 7)).

increase under the optimal monetary-fiscal policy mix in a liquidity trap. This rise in taxes, is aimed, inter-alia, at creating “fiscal space” for a future expansionary fiscal policy in the form of a tax cut once the demand disturbance has dissipated. The commitment to this loose fiscal policy is accompanied by a commitment to a loose monetary policy, both policies being used to boost demand while the economy is in the liquidity trap. Our analysis, however, further shows that, as the economy comes closer to its fiscal limit, a marginal rise in taxes creates less “fiscal space”, and hence the policymaker gradually loses its fiscal ammunition to engineer this future boom. The significant increase in welfare losses is also due to the lower ability of the policymaker to correct for the initial (wider) long-run allocation distortions, which is also linked to the very same loss in “fiscal ammunition”.

The second insight concerns positive technology shocks. Relatively to demand shocks, these shocks shift upwards the short-run “Laffer curve” and allow the policymaker to attain a better welfare outcome than for demand shocks in the proximity of the FL. Specifically, the character of optimal policy in a liquidity trap resembles the one in response to demand shocks: the policymaker initially rises taxes and promises to conduct both expansionary monetary and fiscal policy after the shock. In contrast to demand shocks however, even when the economy is initially at fiscal limit, because of the enhanced production capacity, the initial optimal rise in taxes results in larger tax revenues, and hence in additional fiscal space that can be used to engineer the future boom and to permanently correct long-run distorsions.

Hereafter, the paper is organized as follows. Section 2 discusses the related literature. Section 3 describes the model. Section 4 describes the calibration. Section 5 analyses the long-run steady-state “Laffer curve” of the model and defines the long-run fiscal limit. Section 6 derives dynamics under the optimal policy when the ZLB is binding– first for a demand shock (subsection 6.1) and then for a technology shock (subsection 6.2). Section 7 concludes by summarizing main findings and discussing future extensions.

2 Related Literature

The paper contributes to two different strands of literature. One is optimal business-cycle policy. In this literature, some previous normative analyses considered optimal monetary policy in an environment in which a lump-sum tax is available. Some of these analyses abstracted from the existence of a ZLB (e.g. Clarida, Gali and Gertler (1999), Benigno and Woodford (2005)), while others took it under consideration (e.g. Eggertsson and Woodford (2003)). A complementary set of papers looked instead at the case of optimal monetary-fiscal policy with distorsionary taxation. And again, some of them did not consider the ZLB constraint (e.g. Benigno and Woodford (2003), Schmitt-Grohe and Uribe (2001), Correia et al (2001), Siu (2001)), while others did (e.g. Eggertsson and Woodford (2006), Schmidt (2013), Nakata (2016)). In a companion paper, we studied how the convergence to the fiscal limit affected optimal monetary-fiscal policy in the absence of the ZLB constraint. We now contribute to this strand of New-Keynesian literature by investigating the same question, but focusing instead on the case where the ZLB is binding under optimal policy.

The second strand of literature to which we contribute is the "fiscal limit" literature (e.g. Bianchi and Melosi (2018), Leeper (2013), Davig, Leeper, and Walker (2011), Leeper and Walker (2011)). Papers in this literature have warned against the dire consequences of the fiscal limit for the conduct monetary policy, focusing on the case where excessive government commitments of the private sector render public debt unsustainable because of the existence of a fiscal limit. Our paper contributes to this literature by pointing out that the fiscal limit has implications for the optimal business cycle policy even before the long-run public debt becomes unsustainable.

3 The Model

The analytical framework is an extension of the cashless-limit basic New Keynesian (NK) model (chapter 3 in Galí (2015) or Woodford (2003)) with an endogenous fiscal limit. In building our analytical setup, we model explicitly the two on-going low-frequency phenomena as affecting the long-run equilibrium of the model. In particular, a lower long-run interest rate is mapped to a lower value of the steady-state interest rate which is pinned down by a higher time discount factor[5]. Furthermore, the relevant fiscal limit in the analysis is the steady-state ("long-run") fiscal limit.

The long-run fiscal limit (FL) is introduced by assuming that steady-state lump-sum transfers to households (which proxy for pre-committed pensions and healthcare expenditures for the elderly) are financed with distorsionary taxes[6]. For simplicity, we shall assume that only labor income taxes are available. Distorsionary taxation generates a "Laffer curve" in equilibrium, with its peak in

[5]The same approach is followed as well in the ZLB literature.

[6]In modeling the fiscal pressures due to aging populations (depicted in figure 1) as higher transfers to households, we follow the fiscal-limit literature (e.g. Bi (2012), Davig, Leeper, and Walker (2010)).

steady-state defining the (long-run) "fiscal limit". Larger transfers push the economy closer to its fiscal limit. Government transfers are treated as exogenously given.

The model is populated by a large number of identical households, a continuum of size one of monopolistic firms, and a consolidated monetary-fiscal authority.

3.1 Households

The representative household decides each period how much to consume C_t, to work L_t and to invest in one-period riskless nominal public bonds B_t in order to maximize expected inter-temporal lifetime utility

$$E_0\Big\{\sum_{t=0}^{\infty}\beta^t Z_t\Big(\frac{C_t^{1-\sigma}-1}{1-\sigma}-\frac{L_t^{1+\varphi}}{1+\varphi}\Big)\Big\} \tag{1}$$

subject to the sequence of budget constraints

$$P_tC_t + Q_tB_t \leq B_{t-1} + (1-\tau_t)W_tL_t + Div_t + P_t\mathscr{T}, \tag{2}$$

and solvency (transversality) conditions

$$\lim_{T\to\infty} E_t\Big\{\beta^{T-t}\frac{Z_TC_T^{-\sigma}B_T}{Z_tC_t^{-\sigma}P_T}\Big\} \geq 0 \tag{3}$$

for $t = 0, 1, 2, ...,$ where $C_t \equiv [\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}}di]^{\frac{\varepsilon}{\varepsilon-1}}$ is a standard Dixit-Stiglitz consumption index of differentiated goods with ε a measure of substitutability among them, $P_t \equiv [\int_0^1 P_t(i)^{1-\varepsilon}di]^{\frac{1}{1-\varepsilon}}$ is the unit price of the consumption basket, W_t stands for the nominal wage, B_t represents purchases of one-period public bonds sold at price Q_t which pay one nominal unit at $t+1$, τ_t is a (time-varying) tax rate on labor income, Div_t are dividends from the ownership of firms, $\mathscr{T}$ are real transfers from the public sector, and $log(Z_t) = \rho_z log(Z_{t-1}) + \varepsilon_t^z$ is a demand preference shock. The optimality conditions describing household's behaviour are the sequence of consumption/saving decisions

$$Q_t = \beta E_t\Big\{\frac{C_{t+1}^{-\sigma}Z_{t+1}}{C_t^{-\sigma}Z_t}\frac{P_t}{P_{t+1}}\Big\}, \tag{4}$$

labor supply ones

$$C_t^{\sigma}L_t^{\varphi} = (1-\tau_t)\frac{W_t}{P_t}, \tag{5}$$

together with the solvency conditions (3), and the period budget constraints (2) for $t = 0, 1, 2....$

Up to a first order log-linear approximation, household's optimality conditions write

$$\widehat{c}_t = E_t\{\widehat{c}_{t+1}\} - \frac{1}{\sigma}(\widehat{i}_t - E_t\{\pi_{t+1}\}) + \frac{1}{\sigma}(1-\rho_z)z_t, \tag{6}$$

$$\widehat{\omega}_t = \sigma\widehat{c}_t + \varphi\widehat{l}_t + \frac{\tau}{1-\tau}\widehat{\tau}_t, \tag{7}$$

together with the solvency conditions (3) for $t = 0, 1, 2, ...$, where $\widehat{i}_t \equiv -log(Q_t) - (-log(Q))$ is the deviation from steady-state of the nominal bond yield, $\omega_t \equiv log(\frac{W_t}{P_t}) - log(\frac{W}{P})$ is the log of the real wage rate. The budget constraints are always satisfied in equilibrium (2) given "Walras-Law".

3.2 Firms

Firms are in monopolistic competition and each of them produces a different variety i. They are all endowed with an identical Cobb-Douglas production technology

$$Y_t(i) = A_t L_t^{1-\alpha}(i), \;\; \log(A_t) = \rho_a \log(A_{t-1}) + \varepsilon_t^a \tag{8}$$

and face Calvo-type price adjustment constraints. In this environment, at each date, only $1-\theta$ of them reset their prices according to the optimality condition

$$p_t^* = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t\{\psi_{t+k/t}\} \quad \forall t, \tag{9}$$

where p_t^* is the log of the optimal price, μ is the log of the desired gross markup $\mathscr{M} \equiv \frac{\epsilon}{\epsilon-1}$, $\psi_{t+k/t} \equiv -log(1-\alpha) + (\omega_t + p_t) + l_{t+k/t} - y_{t+k/t}$ is the log of the nominal marginal cost at date $t+k$ of a firm which last reset its price at date t (in logs). The remaining fraction θ of firms satiates its demand at the posted price. A detailed derivation of the price setting equation can be found for instance in Chapter 3 in Galí (2015) or Woodford (2003).

3.3 Public sector

A consolidated monetary-fiscal authority choses optimally the evolution of nominal bond yields $i_t \equiv -\log Q_t$ and of the labor income tax rates τ_t, given pre-committed (exogenous) real period (lump-sum) transfers $\mathscr{T}$, subject to the sequence of flow budget constraints

$$Q_t B_t = B_{t-1} + (P_t\mathscr{T} - \tau_t W_t L_t), \tag{10}$$

and of solvency conditions

$$\lim_{T\to\infty} \beta^T E_t\{Q_T B_T\} = 0 \tag{11}$$

for $t = 0, 1, 2, ...$ The flow budget constraint states that newly issued public debt at each date equals outstanding debt plus the current fiscal deficit. The solvency constraint states that government debt cannot increase at a rate faster than the interest rate. The policymaker can credibly promise to repay its (nominal) debt every period and can also commit to future policy actions. Furthermore, the public authority stands ready to issue non-interest bearing cash-balances, implying that the ZLB

$$i_t \geq 0 \tag{12}$$

is a constraint on the set of possible equilibria that can be achieved by the policymaker.

Hereafter, we will use in the analysis the log-linearized versions of the two constraints around the non-stochastic zero-inflation steady-state

$$\widehat{b}_t = \beta^{-1}\Big(\widehat{b}_{t-1} - \pi_t - \frac{\tau W/PL}{B/P}(\widehat{\tau}_t + \widehat{\omega}_t + \widehat{l}_t)\Big) + \widehat{i}_t \quad \forall t, \tag{13}$$

$$\lim_{T\to\infty} \beta^T E_t\{\widehat{b}_T + \widehat{p}_T - \widehat{i}_T\} = 0 \tag{14}$$

where $\widehat{b}_t \equiv log(\frac{B_t}{P_t}) - log(\frac{B}{P})$, together with the constraint on the nominal public bond yield

$$\widehat{i}_t + \rho \geq 0 \;\; \forall t \tag{15}$$

where $\rho \equiv -log(Q) = -log(\beta)$.

3.4 Market clearing

Market clearing conditions are imposed for each variety i, namely $Y_t(i) = C_t(i)$, and they imply in aggregate (in log-deviations from steady-state):

$$\widehat{y}_t = \widehat{c}_t \tag{16}$$

where $\widehat{y}_t = log(Y_t) - log(Y)$ with $Y_t \equiv [\int_0^1 Y_t(i)^{1-\frac{1}{\varepsilon}} di]^{\frac{\varepsilon}{\varepsilon-1}}$. Aggregate price dynamics follow the same path as in the standard basic NK model (e.g. Chapter 3 in Galí (2015)), namely

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \lambda \widehat{mc}_t \tag{17}$$

with $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta} \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$, where $\widehat{mc}_t \equiv (\psi_t - p_t) - (-\mu) = \widehat{\omega}_t - (a_t - \alpha \widehat{l}_t)$ is the log-deviation from steady-state of the average real marginal cost.

The market clearing condition on the labor market writes

$$L_t = \int_0^1 L_t(i) di,$$

and implies up to a first order log-linear approximation

$$(1-\alpha)\widehat{l}_t = \widehat{y}_t - a_t \tag{18}$$

On the bond market, government supply equals household demand.

3.5 Equilibrium

Combining representative household's consumption/saving decision (6) with the goods market clearing condition (16), we can summarize aggregate demand by

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} - \frac{1}{\sigma}(\widehat{i}_t - E_t\{\pi_{t+1}\}) + \frac{1}{\sigma}(1-\rho_z)z_t \;\; \forall t \tag{19}$$

Furthermore, aggregate price dynamics (17) combined with the goods market clearing condition (16), the labor market clearing condition (18) and the labor supply (7), can be used to summarize the supply-side of the model economy as follows

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \lambda\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)\widehat{y}_t - \lambda\frac{1+\varphi}{1-\alpha}a_t + \lambda\frac{\tau}{1-\tau}\widehat{\tau}_t \;\; \forall t, \tag{20}$$

Note that (20) defines a short-run aggregate supply relation between inflation and output, given the current tax rate, and expectations regarding future inflation, output and taxes.

Finally, using (7), (18) and (16), we can write the law of motion of end-of period public-debt

(13) also in terms of aggregate variables

$$\widehat{b}_t = \beta^{-1}\Big[\widehat{b}_{t-1} - \pi_t - \Big((1-\beta) + \frac{\bar{t}}{\bar{b}}\Big)\Big(\Big(\sigma + \frac{1+\varphi}{1-\alpha}\Big)\widehat{y}_t - \frac{1+\varphi}{1-\alpha}a_t + \frac{1}{1-\tau}\widehat{\tau}_t\Big)\Big] + \widehat{i}_t \quad \forall t \tag{21}$$

The tax rate τ_t and the policy rate $\widehat{i}_t \geq -\rho$ are both chosen as endogenous responses to macroeconomic developments under the optimal policy mix. The monetary policy instrument i_t can be used to affect directly aggregate demand (19), but only to the extent allowed by the ZLB (15). Because of distortionary taxation, variations in the policy rate have also a direct effect on debt sustainability (21). On top of that, monetary policy has further indirect fiscal consequences since the equilibrium inflation affects the real burden of public debt. The same applies to the equilibrium output which affects the tax base, and hence the equilibrium tax income. The fiscal instrument τ_t can be used to affect directly aggregate supply (20), as well as debt dynamics (21), which it further affects indirectly via the equilibrium levels of inflation and output.

4 Baseline calibration

Our baseline calibration, summarized in Table 1, follows Nakata (2017) which uses a similar setup (but with public expenditures) to study optimal monetary-fiscal policy subject to the ZLB constraint. Specifically, we set $\beta = \frac{1}{1+0.075}$, $\varphi = 1$, $\theta = 0.75$, $\alpha = 0$, $\varepsilon = 10$, $\sigma = 1/6$. The only difference is the assumption of an initial steady-state debt equal to 60 percent of steady-state annual GDP, or 2.4 quarters' GDP (i.e. $\bar{b} = 2.4$), instead of 0.5 quarters' GDP in Nakata (2017). We choose this value because it is more conventional in the NK literature on optimal monetary and fiscal policy (see for instance Benigno and Woodford (2003) or Eggertsson and Woodford (2006), and most recently Galí (2020)). As shown in the following section, this calibration implies a nicely-shaped Laffer curve with a peak attained for a labor income tax rate equal to 50%.

We check the robustness of our results by also assigning parameters the textbook values in Gali (2015). This alternative calibration is summarized as well in Table 1. Even though similar results emerge, we don't use this calibration as the baseline one for exposition purposes. In particular, conditional on the parsimonious tax system considered in the model, its implied long-run Laffer curves is very skewed to the right with the peak attained for an empirically implausible tax rate value of 90% (included in the Appendix 9.3 on page 38).

Table 1: Calibration

Parameter	Description	Baseline	Alternative
β	Discount factor	$\frac{1}{1+0.075}$	0.99
σ	Curvature of consumption utility	$\frac{1}{6}$	1
φ	Curvature of labor disutility	1	5
α	Index of decreasing returns to labor	0	0.25
ε	Elasticity of substitution of goods	10	9
θ	Calvo index of price rigidities	0.75	0.75
$\bar{b}$	Steady-state debt to GDP	2.4	2.4

Note: Values are shown in quarterly rates.

5 Steady-state Laffer curve

Proportional labor income taxes generate an endogenous Laffer-curve steady-state relation between the tax rate τ and tax revenues $\tau\frac{W}{P}L$. Specifically, a higher tax rate increases tax revenues when the current rate is below a certain threshold, but it reduces it when it is above it (figure 4). This is because a rise in the tax rate has two opposing effects on tax revenues. On the one hand, it has a direct positive effect at a given tax base $\frac{W}{P}L$. On the other hand however, it has an indirect negativ effect because it shrinks the tax base. This is because it decreases the after-tax wage (equation (5)) inducing households to work less (L declines), and hence produce less (Y declines) in equilibrium. As a result, labor income $\frac{W}{P}L$, which is a direct function of output, declines[7]. Up to a certain threshold value of the tax level $\bar{\tau}$, the former effect is stronger and a rise in the tax rate increases revenues. Beyond this threshold however, the latter effect prevails, and tax revenues decline.

We call this steady-state revenue-maximizing tax rate *the long-run "fiscal limit"*. As already pointed out, its key feature is that any further increase in the tax rate above this level $\bar{\tau}$ induces a decline in tax revenues in equilibrium. Otherwise stated, this tax rate is associated to the highest level of tax income that can be collected given the structural parameters of the economy. Using the zero-inflation steady-state relation implied by the government flow of funds (10)

$$\bar{\tau}\frac{W}{P}L = \frac{B}{P}\Big(1-\beta\Big) + \mathscr{T}, \tag{22}$$

we can see that this tax rate is also associated to the highest level of risk-free public debt and period

[7] The steady-state wage bill (hence, the tax base) equals $\frac{W}{P}L = (1-\alpha)\mathscr{M}^{-1}Y$.

real transfers that the government can service in equilibrium[8]. If the policymaker were to promise to repay higher such levels, it would not have the means to deliver on its promises. It would have to default, either outright or indirectly via inflation given that public debt is nominal[9].

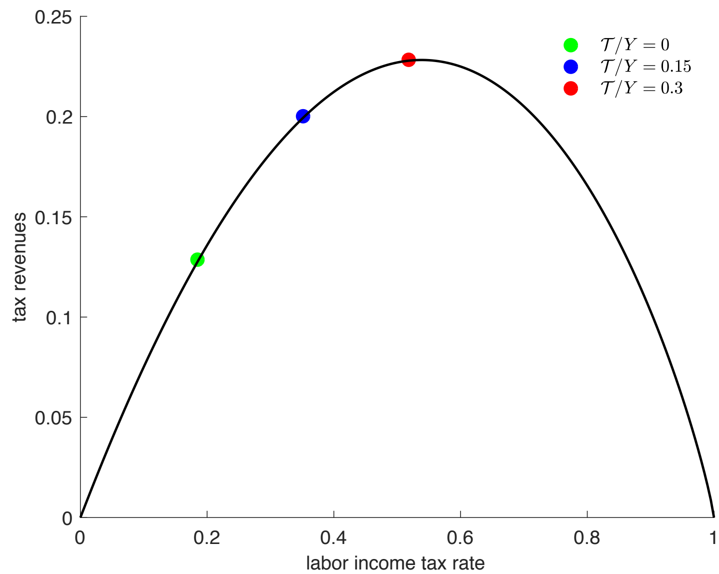

Figure 4: Steady-state Laffer curve

As implied by the government flow of funds (22) above, high stable real long-run transfers $\mathscr{T}$ imply high real tax receipts $\tau\frac{W}{P}L$ needed to finance them. Thus, depending on $\mathscr{T}$, the economy may be far or close to its long-run fiscal limit. Figure 4 depicts the position of the economy on its long-run Laffer curve for three different values of real transfers: 0 (green), medium (blue), high (red), and mentions the associated steady-state transfers-to-output ratio in equilibrium. For a given real (inherited) debt level $\frac{B}{P}$, the higher the promised real long-run transfers, the higher the equilibrium steady-state transfers-to-output ratio, and the closer the economy to its "fiscal limit"[10].

Our welfare analysis focuses on how the proximity of the economy to its long-run fiscal limit (i.e. to the peak of the Laffer curve in figure 4) affects the optimal monetary-fiscal policy response to

[8]Note that an upper bound on the left-hand-side of the equality, implies an upper bound on right-hand-side.

[9]Note that because debt is nominal, its real return can be eroded by inflation:

$$\bar{\tau}\frac{W}{P}L = \Pi^{-1}\frac{B}{P}\Big(1-\beta\Big) + \mathscr{T}$$

[10]Derivations used to generate figure 1 are included in the Appendix on page 37. Note that higher real levels of inherited debt also push economies to their fiscal limit.

negative demand and positive technology disturbances. The latter are assumed to be strong enough such that the ZLB binds[11]. The analysis concerns levels of real transfers that the policymaker can pay in equilibrium. The focus is thus on how promises *a priori* sustainable in the long-run may restrict what optimal policy can achieve at business cycle frequency. We confine attention to *long-run* tax rate levels on the increasing region of the "Laffer curve".

6 Optimal policy

The policymaker chooses the optimal state-contingent path of the tax rate τ_t and of the policy rate i_t so as to maximize the welfare of the representative household, measured by (1), given the exogenously specified evolution of demand Z_t and technology A_t disturbances. When choosing these paths, the policymaker is constrained by the joint evolution of the variables $\{\widehat{y}_t, \pi_t\}$ given by equations (15), (19), (20), (21), (14), each of which must hold for each $t \geq t_0$, given the initial public debt b_{t_0-1}. As in Eggertsson and Woodford (2003), but also Jung, Teranishi and Watanabe (2003), Werning (2011), and most recently Galí (2019), we study optimal policy in the case of fully unanticipated, one-and-for-all shocks (in our case, demand and technology shocks), which bring the natural rate of interest temporarily in negative territory.

We characterize optimal policy from a "timeless perspective" (Woodford (1999; 2003, chapter 7)). Specifically, "from that date forward to which the policymaker would have wished to commit itself at an earlier date". The steady-state distorsions (i.e., the size of discrepency between steady-state output and its efficient counterpart) are large and further widen as the model economy approaches fiscal limit (table 2)[12]. This implies that the effects of stabilization policy on the average level of output matter for the welfare evaluation of alternative policies (see equation 48 on page 41 in the Appendix). Benigno and Woodford (2003) show that it is nonetheless possible to correctly evaluate welfare under alternative policies to second order in a bound on the amplitude of the exogenous disturbances using only a log-linear approximation to the model equilibrium relations to characterize equilibrium dynamics under a given policy. This is possible when one uses as one's welfare measure

[11] Importantly, if taxes were lump-sum, the economy would not face a fiscal limit.

[12] As standard in the NK literature, the degree of inefficiency of the steady-state output level is measured by the steady-state wedge between the marginal rate of substitution between consumption and leisure and the marginal product of labor. As shown in the Appendix on page 38, its expression in our model is given by

$$\Phi \equiv 1 - \frac{1-\tau}{\mathscr{M}}, \quad \Phi \in (0,1) \tag{23}$$

A strictly positive distorsion (i.e. $\Phi > 0$ is associated to a level of steady-state output inefficiently low.

a quadratic loss function in which the effects of stabilization policy on average output have already been taken into account in the loss function, so that the loss function is purely quadratic, rather than depending explicitly on the average level of output. Eggertsson and Woodford (2006) applied this method to study optimal monetary-fiscal policy in a liquidity trap. Since our analysis may be viewed as an extension of their study, we follow the same approach to characterize optimal policy in the presence of both ZLB and FL constraints.

Table 2: Long-run distorsions on the convergence path to the fiscal limit

Transfer-to-output	Distorsion
0	0.2674
0.1	0.3674
0.15	0.4174
0.3	0.5674

As shown in the Appendix 9.5 on page 39, the quadratic approximation to the expected discounted utility of the representative household (1) in our model is a decreasing function of the objective

$$\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\Big(Q_x^{22}\widetilde{y}_t^2+2\xi_t^{\tau}\widehat{\tau}_t+q_{\pi}\pi_t^2\Big) \tag{24}$$

where $\widetilde{y}_t \equiv \widehat{y}_t-\widehat{y}_t^*$ is the welfare relevant output gap with $\widehat{y}_t^* \equiv -\frac{Q_\xi^{21}}{Q_x^{22}}z_t-\frac{Q_\xi^{22}}{Q_x^{22}}a_t$ and $\xi_t^{\tau} \equiv \Big(Q_\xi^{11}-\frac{Q_x^{12}Q_\xi^{21}}{Q_x^{22}}\Big)z_t+\Big(Q_\xi^{12}-\frac{Q_x^{12}Q_\xi^{22}}{Q_x^{22}}\Big)a_t$ (a function of exogenous shocks). Coefficients Q are functions of structural parameters defined in the Appendix 9.5 on page 48. We may rewrite the constraints faced by the benevolent policymaker in terms of $\widetilde{y}_t$ as:

$$(1)\ \ \widetilde{y}_t=E_t\{\widetilde{y}_{t+1}\}-\frac{1}{\sigma}(\widehat{i}_t-E_t\{\pi_{t+1}\})+\frac{1}{\sigma}(1-\rho_z)z_t+E_t\{\Delta\widehat{y}_{t+1}^*\} \tag{25}$$

$$(2)\ \ \pi_t=\beta E_t\{\pi_{t+1}\}+\lambda\Big(\sigma+\frac{\alpha+\varphi}{1-\alpha}\Big)\Big(\widetilde{y}_t+\widehat{y}_t^*\Big)-\lambda\frac{1+\varphi}{1-\alpha}a_t+\lambda\frac{\tau}{1-\tau}\widehat{\tau}_t \tag{26}$$

$$(3)\ \ \widehat{i}_t+\rho\geq 0 \tag{27}$$

$$(4)\ \ \widehat{b}_t=\beta^{-1}\Big[\widehat{b}_{t-1}-\pi_t-\Big((1-\beta)+\frac{\bar{t}}{\bar{b}}\Big)\Big(\Big(\sigma+\frac{1+\varphi}{1-\alpha}\Big)(\widetilde{y}_t+\widehat{y}_t^*)-\frac{1+\varphi}{1-\alpha}a_t+\frac{1}{1-\tau}\widehat{\tau}_t\Big)\Big]$$
$$+\widehat{i}_t \tag{28}$$

$$(5)\ \ \lim_{T\to\infty}\beta^T E_t\{\widehat{b}_T+\widehat{p}_T-\widehat{i}_T\}=0, \tag{29}$$

$$(6)\ \ \widehat{p}_t=\pi_t+\widehat{p}_{t-1} \tag{30}$$

for $t = 0, 1, 2....$ Constraint (25) is the familiar "Dynamic IS-equation", whereas (26) is the "New-Keynesian Philips curve" relation extended to take account of the effects of variations in taxes on supply costs. Note that the tax instrument generally has a "cost-push" effect. Specifically, with the exception of one particular case where the variation in the output target is most favorable, any variation in the tax rate precludes simultaneous stabilization of inflation and the welfare relevant output gap. Equation (27) is the ZLB constraint on the policy rate, while (28) is the flow-budget constraint which together with the government solvency condition (29) further restrains the possible equilibrium paths that can be attained under optimal policy. Complete stabilization of inflation, output gap and taxes (which would minimize the welfare criterion) is generally inconsistent with the constraints (25) to (30) above. Thus, the policymaker will strive to find the policy which yields the best welfare trade-off for the representative household.

We consider the problem of choosing state-contingent paths $\{\pi_t, \widetilde{y}_t, \widehat{i}_t \, \widehat{\tau}_t, \widehat{b}_t\}$ to minimize (24) subject to the constraints (25) to (30), given an initial public debt b_{-1}. We assume the economy is in steady-state prior to the shock. The Lagrangian method gives the following optimality conditions describing the dynamics of the economy under optimal policy

$$\widetilde{y}_t: \; Q_x^{22}\widetilde{y}_t + \lambda_t^1 - \beta^{-1}\lambda_{t-1}^1 - \lambda\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)\lambda_t^2 + \beta^{-1}\left[(1-\beta) + \frac{\bar{t}}{\bar{b}}\right]\left(\sigma + \frac{1+\varphi}{1-\alpha}\right)\lambda_t^4 = 0$$

$$\pi_t: \; q_\pi \pi_t - \frac{1}{\sigma\beta}\lambda_{t-1}^1 + \lambda_t^2 - \lambda_{t-1}^2 + \beta^{-1}\lambda_t^4 - \lambda_t^6 = 0$$

$$\widehat{i}_t: \; \frac{1}{\sigma}\lambda_t^1 + \lambda_t^3 - \lambda_t^4 = 0, \;\; t \leq T-1, \;\; \frac{1}{\sigma}\lambda_T^1 + \lambda_T^3 - \lambda_T^4 - \lambda_T^5 = 0,$$

$$\widehat{b}_t: \; \lambda_t^4 - \lambda_{t+1}^4 = 0, \;\; t \leq T-1, \;\; \lambda_T^4 + \lambda_T^5 = 0,$$

$$\widehat{\tau}_t: \; \xi_t^\tau - \lambda\frac{\tau}{1-\tau}\lambda_t^2 + \frac{\beta^{-1}}{1-\tau}\left[(1-\beta) + \frac{\bar{t}}{\bar{b}}\right]\lambda_t^4 = 0,$$

$$\widehat{p}_t: \; \lambda_t^6 - \beta\lambda_{t+1}^6 = 0, \;\; t \leq T-1, \;\; \lambda_T^5 + \lambda_T^6 = 0,$$

$$\lambda_t^3(\widehat{i}_t + \rho) = 0, \; \widehat{i}_t + \rho \geq 0, \; \lambda_t^3 \leq 0,$$

for $t = 0, 1, 2, ...$, together with the sequence of equality constraints above. λ_t^i are the Lagrangian multipliers associated to each of the six constraints at time t.

6.1 Demand shocks

6.1.1 Optimal policy in a "liquidity trap"

Figure 5 displays the economy's state contingent evolution in response to a transitory negative demand shock. The occurrence of the disturbance causes the ZLB to bind, and the policymaker is unable to prevent deflation and a negative output gap on impact. The transfers–to–GDP ratio is set to 15% which places the economy relatively close, but still away from its fiscal limit (blue dot on the "Laffer curve" in figure 4).

Optimal policy replicates the pattern identified in Eggertsson and Woodford (2006)[13]. In particular, both monetary and fiscal policies are *history dependent*, in the sense that even after the effects of the shock have dissipated, the policy rate and the tax rate take both temporarily values different from what their eventual long-run values will be. Specifically, optimal policy involves a commitment to keep nominal interest rates and tax rates low for a (limited) period of time after the natural rate returns to its normal level, and this despite a strong output boom (i.e. a positive output gap) and overshooting inflation. The aggregate demand schedule of the economy described by equation (25), and copied for convenience below, shows that since agents are forward-looking, output can be propped up (more specifically, consumption can be encouraged) at a time where monetary ammunition is waining (namely, when the policy rate cannot be declined enough to offset the effects of the negative shock), by promising "future output or/and inflation boom(s)"

$$\uparrow\uparrow y_t = -\frac{1}{\sigma}\Big(\widehat{i}_t - \overbrace{\uparrow E_t\{\pi_{t+1}\}}^{\text{promised monetary expansion}}\Big) + \underbrace{\uparrow E_t\{\widehat{y}_{t+1}\}}_{\substack{\text{promised fiscal expansion}\\ \text{promised monetary expansion}}} + \underbrace{\frac{1}{\sigma}(1-\rho_z)z_t}_{<0}$$

The promised expansionary monetary and fiscal policies after the effects of the disturbance have dissipated are activating exactly this type of mechanisms. Subsequently, the distorsions created by the binding ZLB on interest rates when the shock occurs are mitigated through a commitment to use both monetary and fiscal stimulus to create "boom" conditions once the shock has ended. Figure 6 shows that the degree of history dependence (and hence, the duration of the "promissed boom") increases in the duration of the shock. Specifically, while for a one period shock monetary and fiscal policies remain loose for one more period after the shock has dissipated, for a six period

[13] In contrast to our analysis however where taxes are collected on labor income, Eggertsson and Woodford (2006) features instead a VAT tax, which similar to our case, also affects the supply-side of the economy.

shock, they remain loose for five more periods (and hence, for a eleven-period in total).

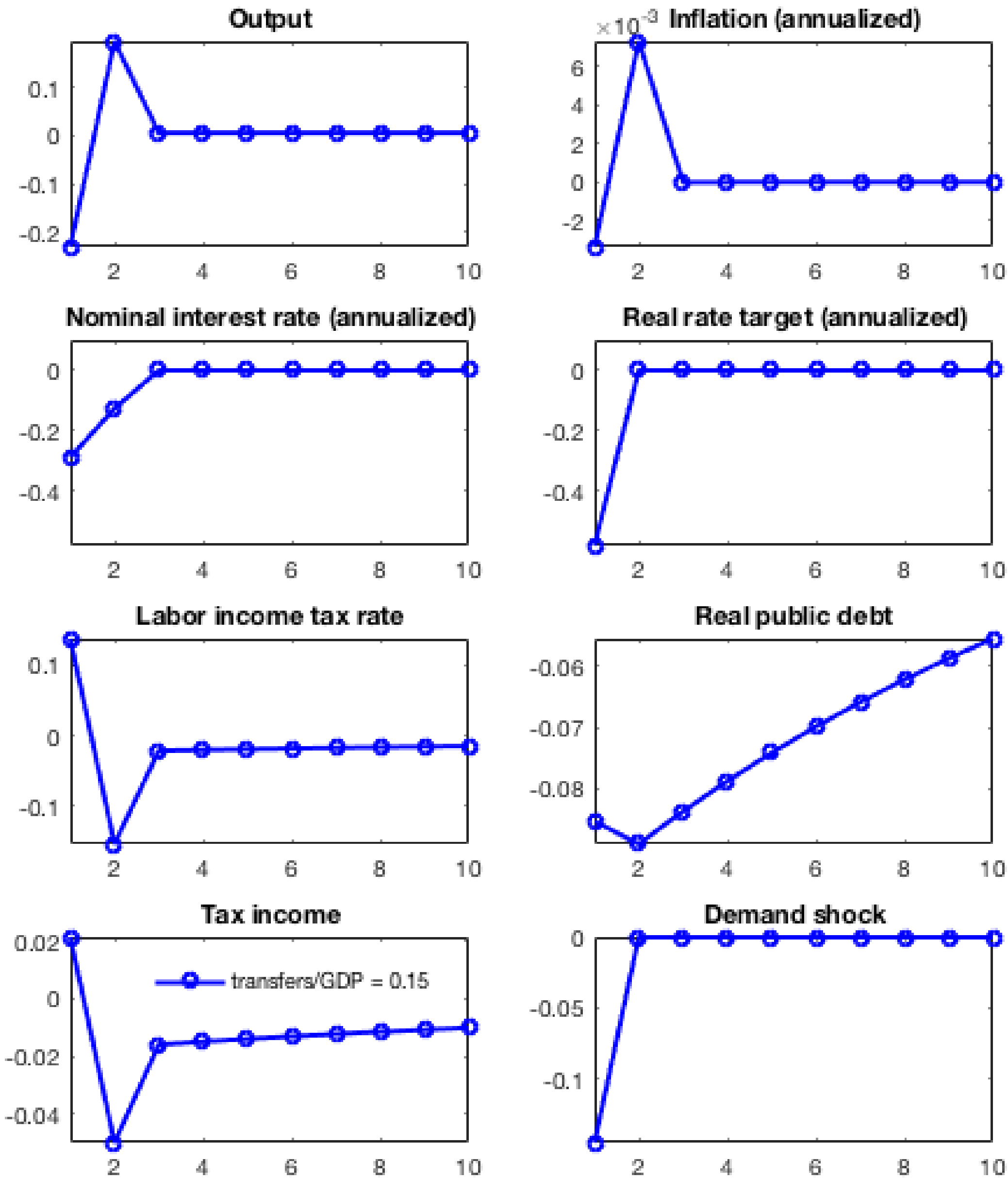

Figure 5: Responses to a transitory demand shock under optimal policy

Note: Y-axis: log- deviation from steady-state for output, inflation, public debt and the demand shock, deviations from steady-state for the interest rate, the efficient rate, tax rate and tax income. X-axis: quarters after the shock

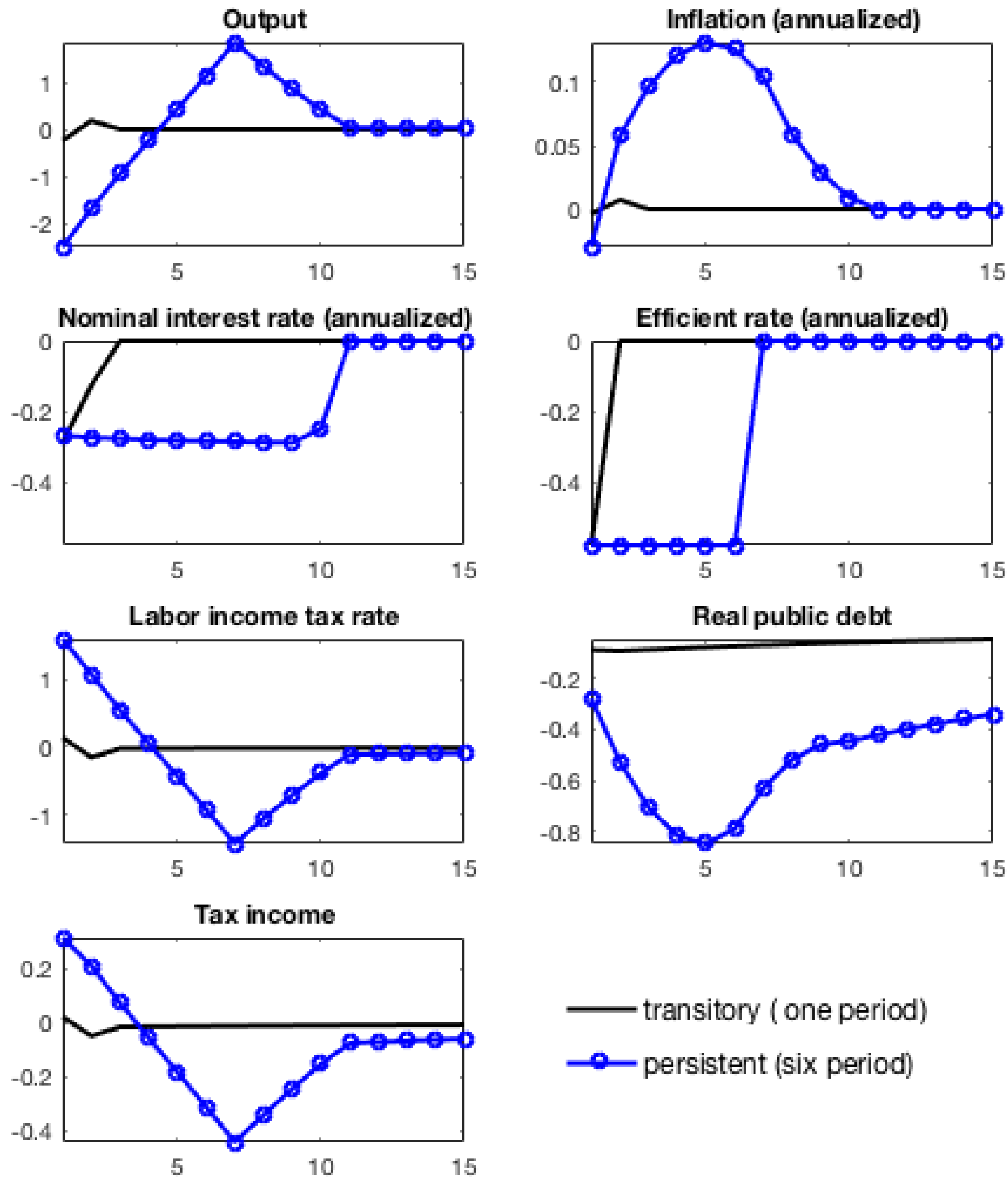

Figure 6: Responses to a demand shock under optimal policy

<u>Note:</u> Y-axis: log- deviation from steady-state for output, inflation, public debt and the demand shock, deviations from steady-state for the interest rate, the efficient rate, tax rate and tax income. X-axis: quarters after the shock

Also, as in Eggertsson and Woodford (2006), the "promised" fiscal stimulus in the period immediately following the return of the efficient rate to its steady-state level is financed by a rise in the tax rate while the shock is active. The tax rise while the economy is in the liquidity trap also helps more directly by providing a cost-push incentive for prices not to be cut. Subsequently, the optimal tax response is to raise taxes in the event of a strong negative demand disturbance driving the policy rate to the ZLB, while committing to lower them during the inflationary boom (that monetary policy will also facilitate once the natural rate of interest returns to a normal level).

Last but not least, as in the latter reference, the policymaker permanently lowers both the tax rate and the public debt under optimal policy. As a result, the terminal steady-state value of output is different than the initial one, implying that optimal policy affects allocations not only in the short run, but also in the long-run. Figures 5 and 6 show that it achieves this both via the initial rise in taxes and via the inflationary boom engineered in the aftermath of the shock. Importantly, this result does not reflect an incentive to inflate away the value of nominal public debt *ex-post* that "a policymaker would instead wish to commit itself in advance not to yield to". This is because the responses shown in figures 5 and 6 represent the responses to a real disturbance that lowers the natural rate of interest under the continuation of an optimal state-contingent commitment that would have been chosen prior to quarter zero, and prior to learning that the disturbance would occur (given the "timeless perspective" of optimal policy).

6.1.2 Fiscal component of optimal policy

Thus far, we have showed that under optimal policy the policymaker is able both to correct long-run distorsions, and to mitigate the underutilization of the productive capacity in the short-run. In this section we have a closer look at the role played by fiscal policy on both these two dimensions.

Fiscal policy affects the equilibrium allocation in a liquidity trap via two channels. One is the "cost-push channel" which works via the aggregate supply schedule of the economy (described by equation (26)). Specifically, the rise taxes at the ZLB increases marginal cost, and hence decreases the aggregate supply schedule of the economy

$$AS: \uparrow\pi_t = \beta E_t\{\pi_{t+1}\} + \lambda\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)\widehat{y}_t + \underbrace{\lambda\frac{\tau}{1-\tau}\widehat{\tau}_t \uparrow}_{\substack{\text{rise in marginal cost} \\ \text{"cost-push" channel}}}$$

exerting direct positive pressures on price inflation. The positive pressures on prices via this "cost-push" channel are used to countervail deflation while the ZLB is binding. Figure 7 shows that the positive effects of the tax hike on equilibrium price inflation via this channel– namely, it shows how the decrease in the aggregate supply curve from AS^o (black solid line) to AS^1 (red dotted line) pushes, at given demand, price inflation upwards.

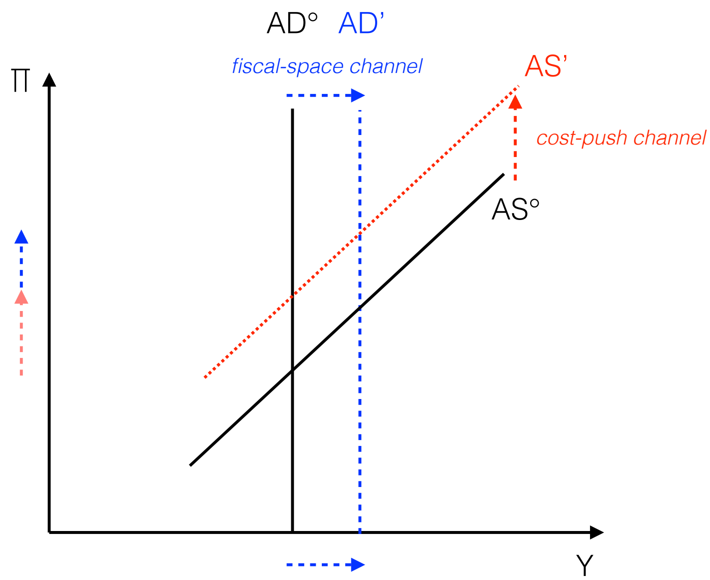

Figure 7: Effects of $\Delta\tau_t > 0$ in a liquidity trap

The other channel works via the aggregate demand channel of the economy. Specifically, the same rise in taxes pushes upwards the demand schedule (described by equation (25)) at the ZLB, by creating the expectation of a future "output boom":

$$AD : \uparrow y_t = \underbrace{\uparrow E_t\{\widehat{y}_{t+1}\}}_{\substack{\text{promised fiscal expansion} \\ \text{“fiscal-space” channel}}} - \frac{1}{\sigma}(\widehat{i}_t - E_t\{\pi_{t+1}\}) + \frac{1}{\sigma}(1-\rho_z)z_t$$

This is because the rise in taxes lowers public debt (or builds up government assets) and creates "fiscal space" (as shown in the left bottom panel of figure 5). The latter is used, together with the extended loose monetary policy, to engineer an output boom after the shock. Figure 7 shows that the positive effects of the tax hike on equilibrium output via this channel– namely, it shows how the increase in the aggregate demand schedule from AD^o (black solid line) to AD^1 (blue dotted line) pushes in equilibrium both output and inflation upwards at a given supply. As a result, in a "liquidity trap", fiscal policy has a joint positive effect on output via the "fiscal–space channel", and on inflation via the“fiscal–space channel” and the “cost–push channel” (via an upward shift in aggregate demand, and a downward shift in aggregate supply, respectively).

Apart from helping stabilize the economy in the face of demand shocks in the short-run, fiscal policy also contributes to the correction of long-run distorsions. In particular, the “fiscal space” created by the initial rise in taxes is further directly used to engineer a permanent decline in real debt, and hence in the final long-run level of taxes required to service it[14]. Given the permanent decline in taxes, the policymaker thus allows the economy to reach a new long-run level of output higher than the one prior to the shock.

6.1.3 Policy problem and the convergence to fiscal limit

The convergence of the economy to its long-run fiscal limit affects the policy design problem on three important dimensions. First, long-run distorsions widen on this convergence path, implying that the overall scope for policy increases. This pattern is showed in table 2. Second, a marginal tax hike decreases more strongly the aggregate supply schedule of the economy. This is because when taxes are high, a further rise is associated to a stronger marginal decline in labor supply and hence to a stronger rise in equilibrium wages. These effects are depicted in the left panel of figure 8. As a result a marginal rise in taxes will have, everything else equal, a stronger positive effect on price inflation (as shown in the right panel of figure 8).

The third important structural change is a weakening of the “fiscal–space” channel, namely a decline in the marginal tax revenues resulted from a given rise in taxes at the ZLB. This pattern is depicted figure 9. This structural change implies that a marginal rise in taxes in the liquidity trap creates less marginal fiscal space for a future output boom (i.e. the “fiscal–space” channel becomes weaker), and thus, it has a smaller positive effect on current output.

To sum up, this section showed that, as the economy convergences to FL, the scope for policy increases (since everything else equal, long-run tax distorsions that need to be corrected become more severe), while policy ammunition, more specifically its fiscal component, may either increase or decrease. Specifically, one the one hand, fiscal ammunition is positively affected via a rise in the sensitivity of price inflation to taxes (i.e. an enhancement of the “cost-push” channel) which, everything else equal, favors inflation stabilization at the ZLB. On the other hand, it is negatively

[14]To these direct effects adjoin the indirect ones via the the equilibrium inflation created via expansionary monetary policy during the boom.

affected by a weakening of the“fiscal space” channel that can be used to prop up output in the liquidity trap.

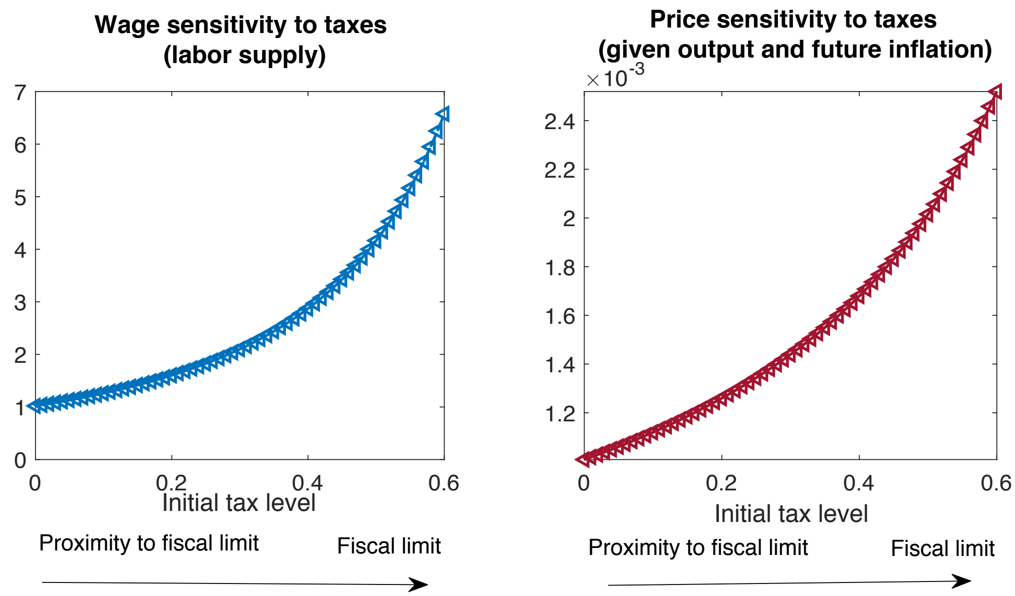

Figure 8: Proximity to fiscal limit and inflation sensitivity to a tax hike

Note: Y-axis: (i) left panel– increase in wage on the labor supply schedule in response to a one percentage point increase in the tax rate, (ii) right panel– increase in price inflation in response to a one percentage point increase in the tax rate at given output and future inflation

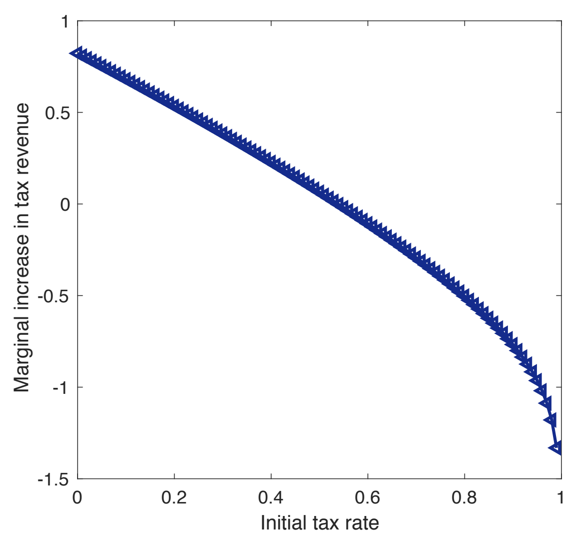

Figure 9: Proximity to FL and marginal tax revenue (steady-state)

Provided the shock is persistent enough, it turns out that the loss in fiscal ammunition overcompensates the advantages of greater price sensitivity to tax variations on the convergence path to the

FL. This can be inferred from figure 10 where, in the case of a transitory shock (depicted on the left panel), welfare losses may decrease under optimal policy as the economy approaches fiscal limit indicating a dominant positive effect of price sensitivity to tax variations in that region, while they continuously increase in the case of a more persistent shock (depicted on right panel). Accordingly, provided the shock is persistent enough, welfare losses significantly increase under optimal policy as the economy converges to its FL.

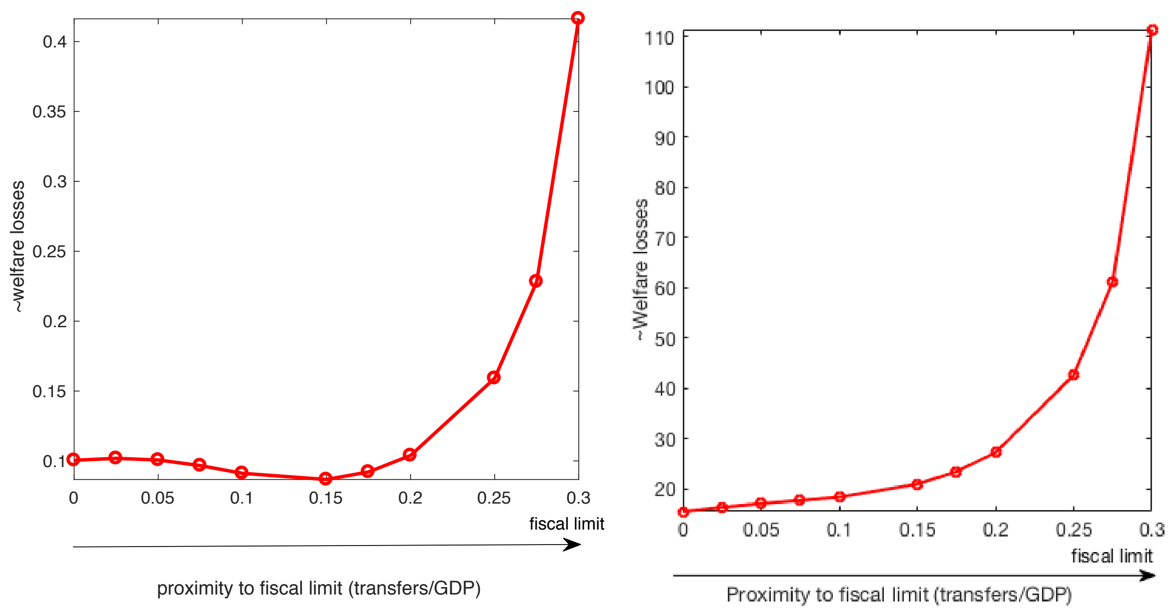

Figure 10: Welfare losses under optimal policy in response to a demand shock as the economy converges to its fiscal limit (transitory shock in the left panel, persistent shock lasting six periods in the right panel)

6.1.4 Responses under optimal policy as the economy approaches fiscal limit

Consistent with previous remarks, figure 11 shows how, for a persistent shock, output is less stabilized in a liquidity trap as the economy comes closer to its FL. Black solid lines stand for zero steady-state transfers, and hence for a position distant from the FL. Blue lines with circles stand for an the intermediate distance from the FL with a steady-state transfer-to-GDP ratio of 15% (already shown in figure 5), while red dashed lines stand for the FL case (which corresponds to a steady-state transfer-to-GDP ratio of 30%). Note also how at the FL the policymaker uses instead more "forward guidance" on monetary policy than when the economy was far away from it, so as to encourage present consumption in the liquidity trap. Propping up output at the ZLB via additional forward guidance is however less effective, and hence current output is less stabilized at the ZLB[15].

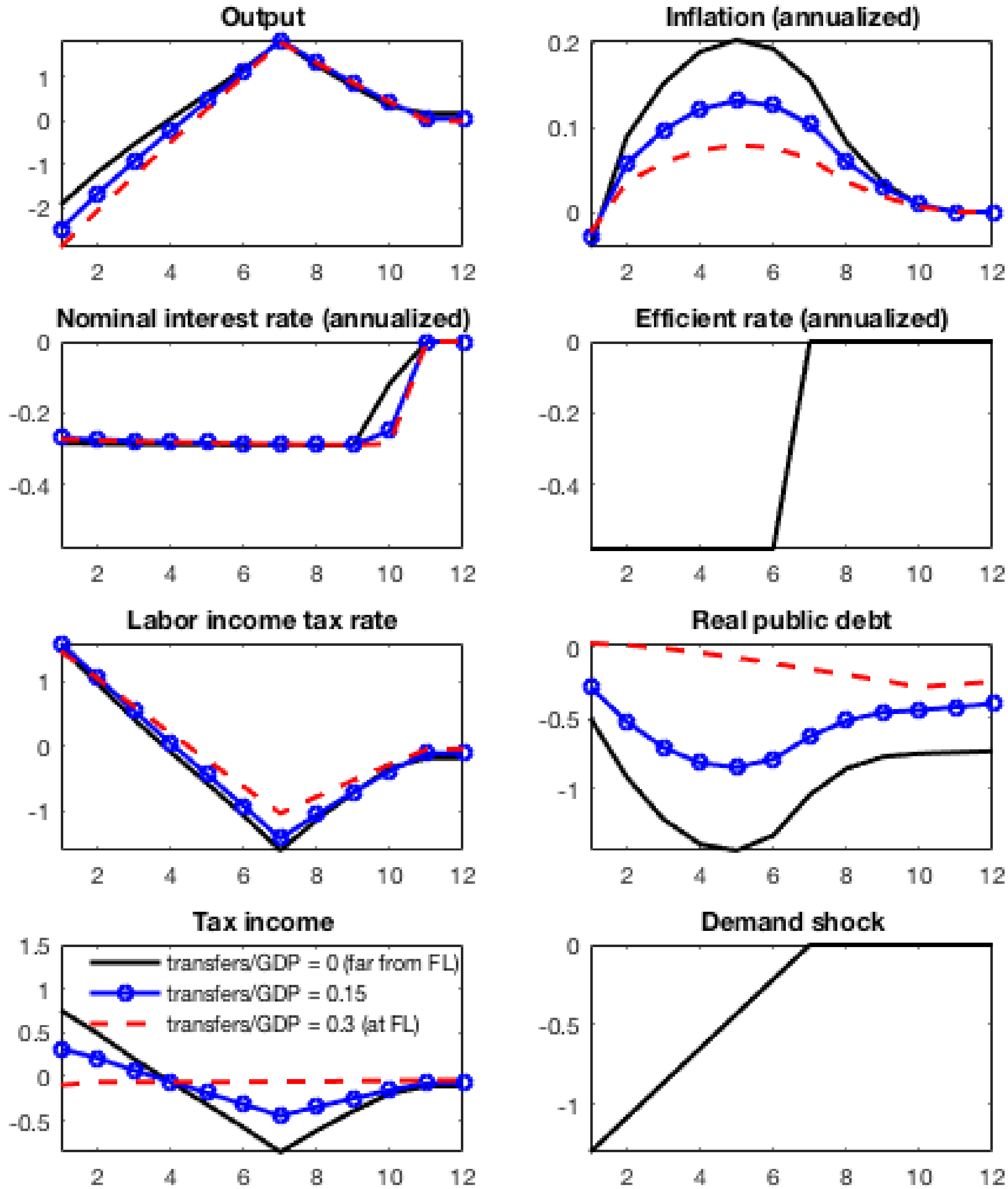

Figure 11: Responses to a persistent demand shock lasting 6 periods under optimal policy

Note: Y-axis: log- deviation from steady-state for output, inflation, public debt and the demand shock, deviations from steady-state for the interest rate, the efficient rate, tax rate and tax income. X-axis: quarters after the shock

[15]One may be puzzled by the decline in "promised" inflation on the convergence path to the FL. This pattern is explained by the higher sensitivity of price inflation to the variation in taxes on this convergence path. Specifically, closer to the fiscal limit, a (future) decline in taxes has stronger negative effects on equilibrium inflation, which counteract the positive ones created via the promised expansionary monetary policy.

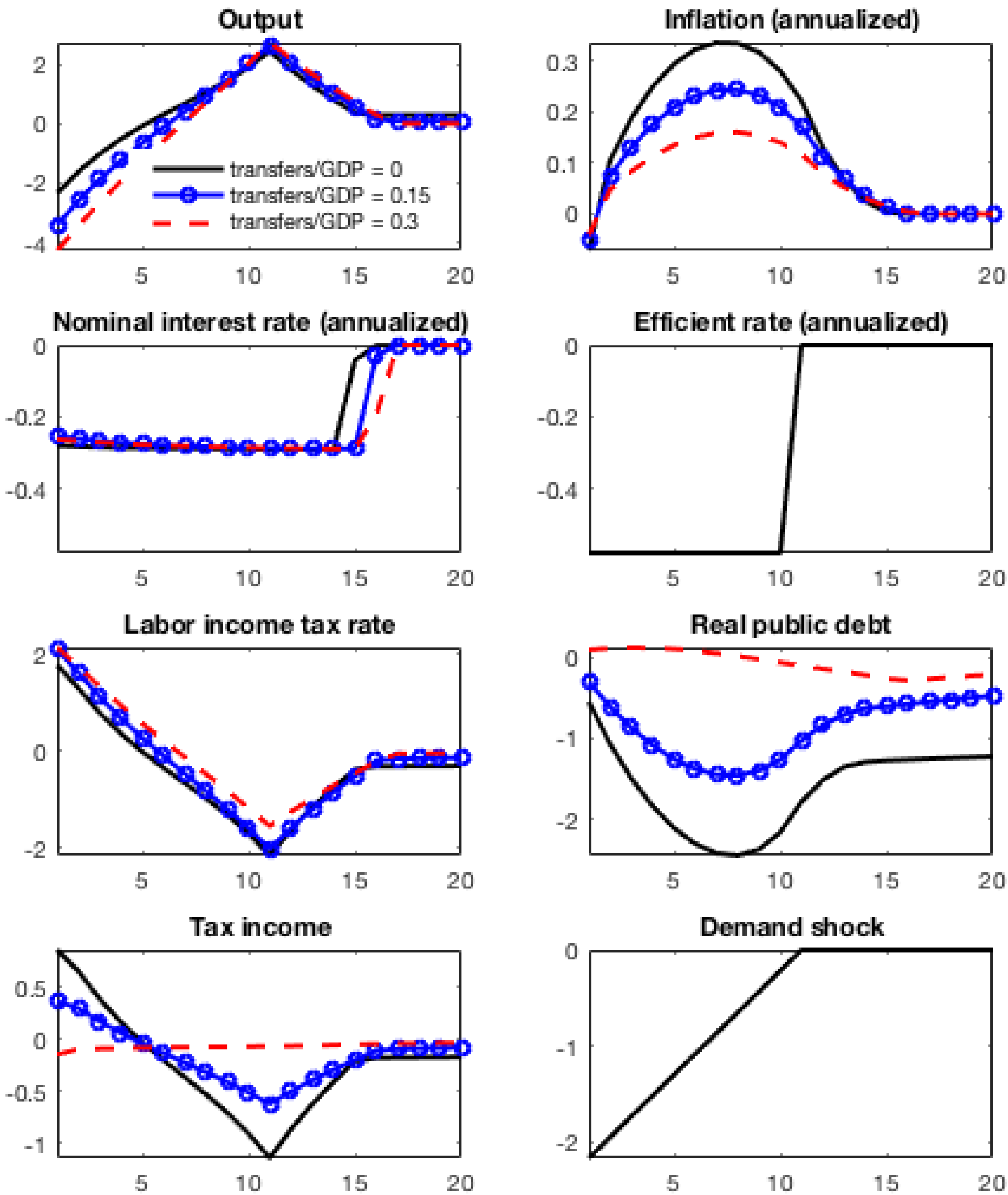

Figure 12: Responses to a persistent demand shock lasting 10 periods under optimal policy

Note: Y-axis: log- deviation from steady-state for output, inflation, public debt and the demand shock, deviations from steady-state for the interest rate, the efficient rate, tax rate and tax income. X-axis: quarters after the shock

The experiment depicted in figure 11 shows also that the loss in fiscal ammunition that can be used to stabilize the economy in the short-run is more generally substituted by a longer span of loose monetary policy after the shock. Specifically, it is not necessary for the economy to have already reached its FL (left panel on the second row). Figure 12 shows that this substitution pattern is even more salient for a more persistent shock lasting ten periods as opposed to six. The gradual substitution of the "fiscal space" channel by "forward guidance" on monetary policy has further adverse implications for the welfare policy outcome not taken into account in our optimal design problem. Specifically, as pointed out in Eggertsson and Woodford (2006), the effectiveness of "forward guidance" on monetary policy is intrinsically linked to the ability of the policymaker to commit to future policy actions, and hence to a history-dependent policy. And this ability may be subject to imperfections in practice. This is however less the case with the "fiscal space" channel. The reason is that the size of the public debt is a state variable that naturally conditions future policy. Hence, current fiscal policy can be used, in effect, to commit future policy to be more expansionary, even if future policy is not conditioned on past developments, i.e. it is purely forward looking. This implies that, in practice, as the economy approaches FL, the ZLB can become an even more significant obstacle to macroeconomic stabilization than our current analysis reveals.

Apart from its harmful effects on the ability of the policymaker to stabilize the economy in the short-run, the gradual loss in fiscal ammunition has also adverse consequences for its ability to correct long-run distorsions. Specifically, figure 13 shows that, as the economy converges to FL, a smaller share of initial long-run distortions are corrected under optimal policy (provided the initial allocation is distorted enough)[16]. Consistently, figure 14 shows that the final long-run allocation remains more distorted in the aftermath of the shock as the economy converges to its FL (provided the initial allocation is distorted enough).

[16]Since the permanent decline in real debt, and hence in long-run taxes, is also due to the prolonged period of below-trend interest rates which implies lower interest expenses, one may think that changes in optimal monetary policy as the economy converges to the FL may also affect the magnitude of the long-run corrections. Note however, that if anything, monetary policy becomes looser as the economy comes closer to its fiscal limit. This is because, as previously showed, it substitutes for expansionary fiscal policy. Thus, we may expect, if anything, the changes in monetary policy to have stronger, and not weaker, negative effects on permanent debt and hence allow for larger corrections. Thus, the lower permanent decline in real debt must necessarily be due to the gradual decline in fiscal space that can be used (i) directly to decrease permanently public debt, and (ii) indirectly, to erode the real value of debt via its effects on the "promised" inflationary boom (since debt is nominal).

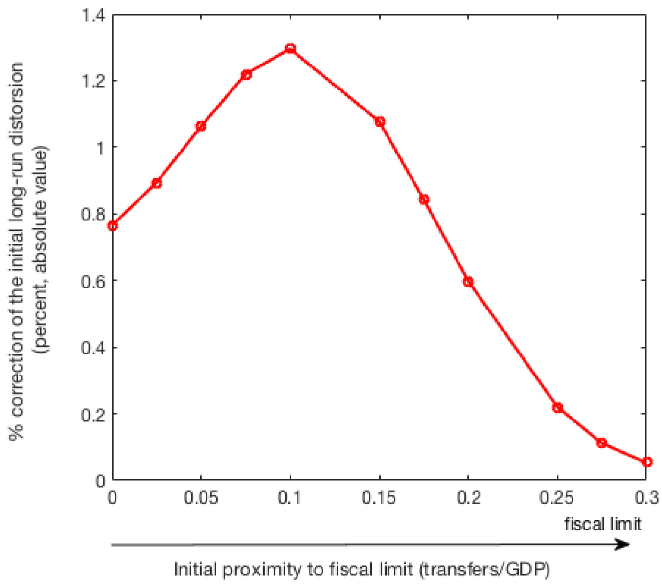

Figure 13: Correction applied to long-run distortion under optimal policy (persistent shock)

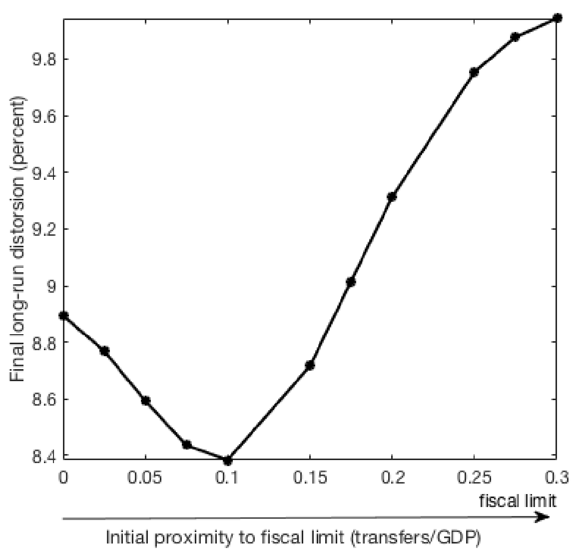

Figure 14: Final-long run distortion under optimal policy after a persistent demand shock

6.2 Technology shocks

Figure 15 shows that for positive technology shocks driving the economy in a liquidity trap, the character of optimal policy is similar to the one for negative demand shocks. For ease of comparison, the magnitude of the technology shock in figure 15 is normalized such that it implies an identical decline in the natural real interest rate as the transitory negative demand shock previously discussed.

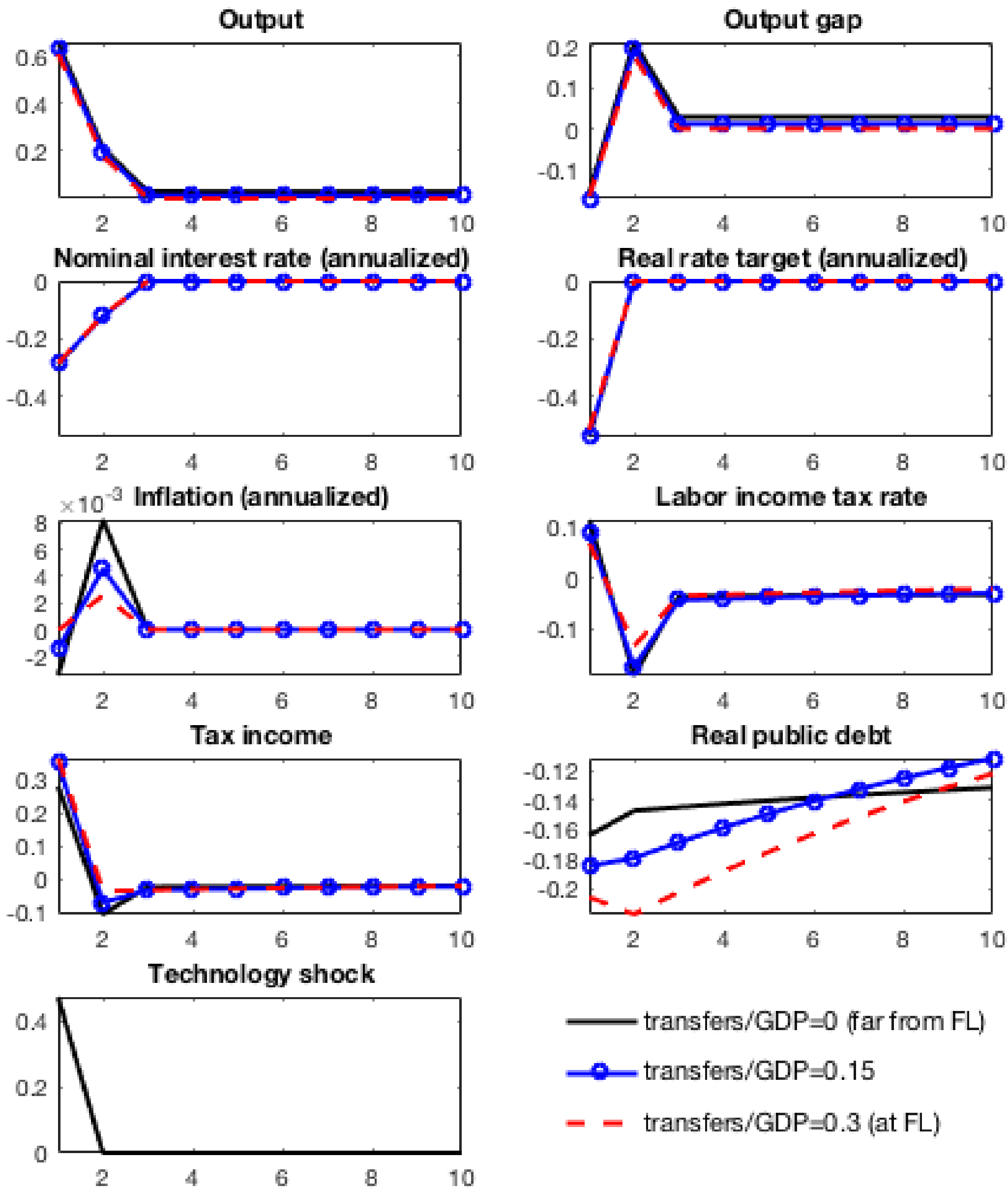

Figure 15: Responses to a transitory technology shock under optimal policy

Note: Y-axis: log- deviation from steady-state for output, inflation, public debt and the demand shock, deviations from steady-state for the interest rate, the efficient rate, tax rate and tax income. X-axis: quarters after the shock

As in the case of demand shocks, when the policy rate is constrained by the ZLB, the policymaker alleviates the effects of waining monetary ammunition by promising to sustain a "future boom" once the real disturbance has dissipated via expansionary monetary and fiscal policy. And, as in the case of demand shocks, taxes rise during the liquidity trap so as to put positive pressures on prices and create fiscal space to finance the future "promised" tax cuts.

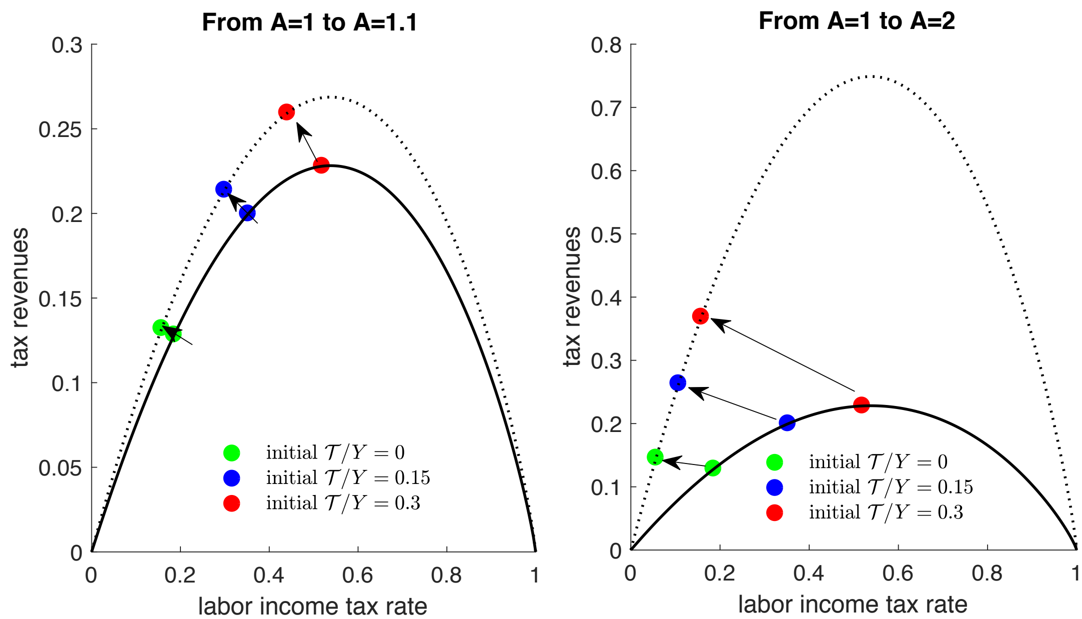

Figure 16: Shift in the steady-state Laffer curve triggered by a rise in the technology parameter A (solid line stands for baseline calibration, whereas the dotted line stands for higher values)

In contrast to demand shocks however, we can observe how, even when the economy is at fiscal limit, the initial rise in taxes results in higher tax revenues, and hence in additional fiscal space that can be used to "promise" the future output boom and to permanently correct long-run distortions. This is because the positive technology shock, in contrast to the negative demand one, shifts upwards the short-run Laffer curve of the economy. Figure 16 gives an intuition on this using the example of steady-state Laffer curves. The figure also shows how the shift depends on the magnitude of the shock: a larger shock induces a larger upward shift (right panel) than a smaller one (left panel). Note however that even if a larger shock allows to create more fiscal space for a future fiscal expansion and for permanent long-run allocation corrections, it also implies that the ZLB constraint is tighter and hence that relatively more fiscal space is needed to mitigate its adverse effects.

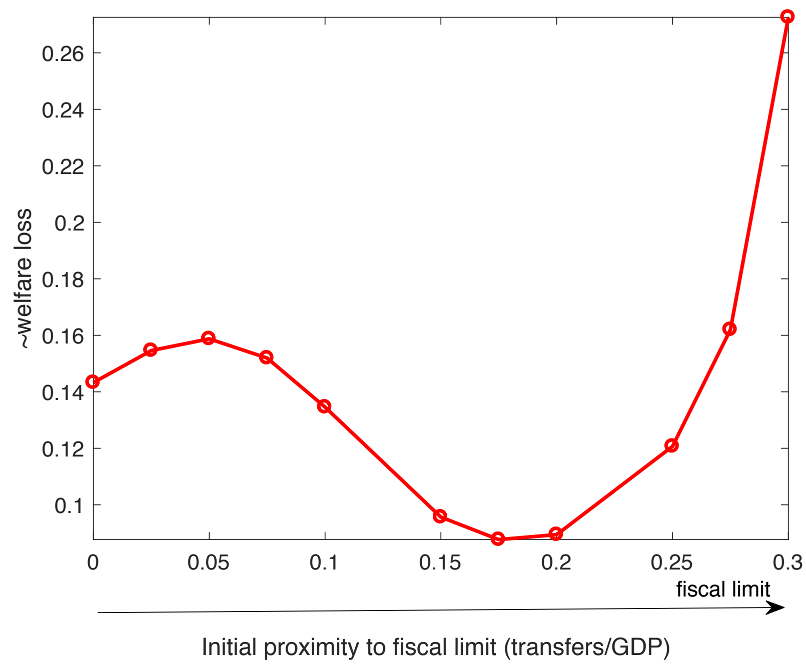

Figure 17: Welfare losses under optimal policy as the economy converges to its fiscal limit – case of transitory technology shock

In the case of the transitory positive technology shock depicted in figure 15, the policymaker gradually chooses to rise taxes less as the economy comes closer to fiscal limit, but that, despite the smaller initial rise in taxes, inflation is better stabilized as the economy converges to its fiscal limit (left panel on the second row). This is explained, as before, by the higher sensitivity of price inflation to taxes when the economy is closer to its FL. This explains the decrease in welfare losses as the economy moves upwards to its fiscal limit for certain initial position on the Laffer curve (figure 17).

Figure 18 shows however that the policymaker chooses to correct a lower share of long-run distorsions despite them being initially wider for initial positions on the Laffer curve relatively close to the "fiscal limit", most likely because of the weakening of the strength of the "fiscal space" channel. As a result, as in the experiment with demand shocks, from a certain position on the Laffer curve onwards, welfare losses increase "exponentially" as the economy approaches its fiscal limit. Importantly however, under optimal policy, welfare losses in response to the transitory positive technology shock (figure 17) are lower than in response to the transitory demand one (left panel in figure 10) in the proximity of the FL[17].

[17]The two shocks can be compared since they have been normalized so as to imply the same decline in the natural

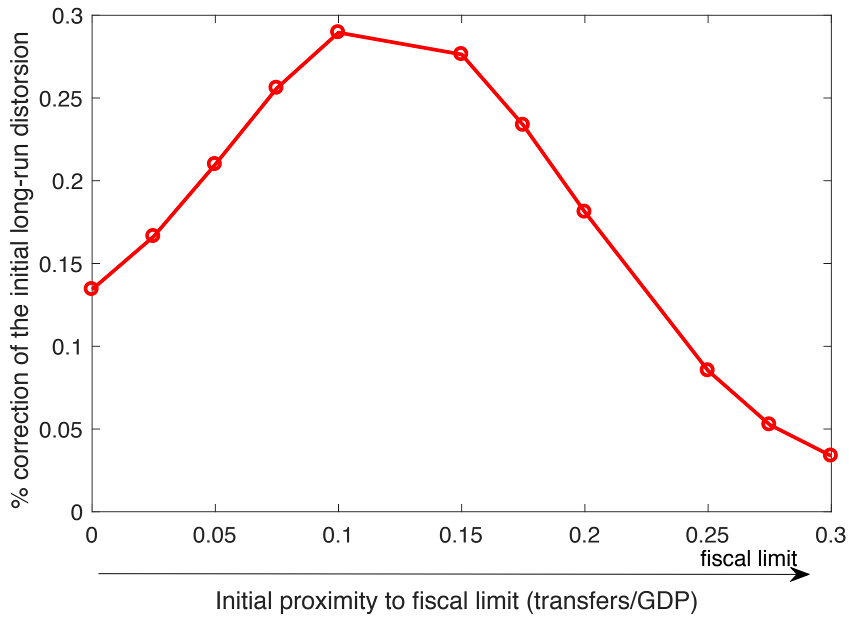

Figure 18: Correction applied to long-run distortion under optimal policy (transitory shock)

7 Concluding Remarks

We studied in this paper how the ZLB and the fiscal limit affect *jointly* the optimal monetary-fiscal response to demand and technology shocks. We used for this purpose an extension of the basic NK model with an endogenous fiscal limit. We abstracted away from both outright default on public debt and outright monetary financing. Our analysis is relevant against the on-going simultaneous decline in long-run interest rates, and convergence to fiscal limit in advanced economies. We mainly contribute to the literature by studying how these two structural trends affect jointly optimal policy.

Our main results are twofold. First, in response to a negative demand shock, as the economy converges to fiscal limit, the reduction in fiscal space restraints the extent of the future boom that the policymaker can promise so as to encourage current consumption while the economy is in the liquidity trap. Subsequently, dynamics become less inflationary, underutlization of productive capacity can be less mitigated and welfare losses increase significantly under optimal policy. Second, in response to positive technology shocks, the short-run Laffer curve shifts upwards. Thus, relatively to demand shocks, the policymaker can attain a better welfare outcome in the proximity of the FL.

Going forward, we want first to study how our results change when we allow for government

rate of interest (i.e. the real rate of interest that the policymaker would have wanted to track in the absence of the ZLB constraint).

expenditures as an additional policy instrument. Another more challenging future step is the analysis of the optimal monetary-fiscal policy mix beyond the fiscal limit. Using our current framework, we can see that, excluding outright default on debt, one available policy option is an increase in the inflation target which would erode the real value of debt and of transfers (if transfers are made in nominal terms instead) up to the point where the peak of the "Laffer curve" is reached again. But would this option be optimal? Namely, would this option which affects all agents in the economy be preferred to outright default on debt which affects only the elderly population (and is detrimental to the sovereign credit record)? And how does this choice depend on the presence of the ZLB which, by itself, may optimally require under certain conditions an increase in the inflation target? And if increasing the inflation target is the joint optimal solution to restore fiscal sustainability and deal with the ZLB, would optimal policy around a zero-inflation steady-state be the same as around a positive one? These are a few of the research questions that we aim to tackle in an extension of our setup in the future.

8 References

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9 Appendix

9.1 Relation between efficient rate and demand preference shocks

I assume the exogenous demand disturbance is of a similar nature as the one in Galí (2015), Section 5.4. Specifically, the efficient rate remains constant to its steady-state level $\rho > 0$ up to (and including) period 0. In period 1 it unexpectedly drops to $-\rho < 0$ (in the case of a negative shock), or it unexpectedly increases to $3\rho < 0$ (in the case of a positive shock), and remains at that level for one period or for six periods. Afterwards it takes again its steady-state value $\rho > 0$. Once the unexpected change in period 1 occurs, the subsequent path of the natural rate is assumed to be known with certainty by all agents.

The efficient rate equals in the model $r_t^e = \rho + z_t - E_t\{z_{t+1}\}$, and hence its deviation from steady-state equals $\widehat{r}_t^e = z_t - E_t\{z_{t+1}\}$.

- In the case of a one period shock $z_t = \widehat{r}_t^e$, where $\widehat{r}_t^e = -\rho - \rho = -2\rho$ for a negative shock, and $\widehat{r}_t^e = 3\rho - \rho = 2\rho$ for a positive shock.

- In the case of a six period shock, for the negative shock $\widehat{r}_6^e = -2\rho$ implies $z_6 = -2\rho$. Furthermore, $\widehat{r}_5^e = z_5 - z_6 = -2\rho$ implies $z_5 = \widehat{r}_5^e + z_6 = -4\rho$. Similarly, $z_4 = -6\rho$, $z_3 = -8\rho$, $z_2 = -10\rho$, $z_1 = -12\rho$.

- In the case of a six period shock, for the positive shock $\widehat{r}_6^e = 2\rho$ implies $z_6 = 2\rho$. Furthermore, $\widehat{r}_5^e = z_5 - z_6 = 2\rho$ implies $z_5 = \widehat{r}_5^e + z_6 = 4\rho$. Similarly, $z_4 = 6\rho$, $z_3 = 8\rho$, $z_2 = 10\rho$, $z_1 = 12\rho$.

9.2 Deterministic steady-state

Household's behaviour is described by

$$C^{\sigma}L^{\varphi} = (1-\tau)\frac{W}{P} \tag{31}$$

$$Q = \beta \tag{32}$$

$$C + Q\frac{B}{P} = Q + (1-\tau)\frac{W}{P}L + \frac{Div}{P} + \mathscr{T}, \tag{33}$$

the one of firms by

$$Y = AL^{1-\alpha} \tag{34}$$

$$1 = \frac{\mathscr{M}}{1-\alpha}\frac{W}{P}\frac{L}{Y} \tag{35}$$

and the flow budget constraint of the public sector by

$$Q\frac{B}{P} = \frac{B}{P} + \left(\mathscr{T} - \tau\frac{W}{P}L\right) \text{ or,} \tag{36}$$

$$Q\bar{b} = \bar{b} + \left(\bar{t} - \tau\frac{W}{P}\frac{L}{Y}\right) \tag{37}$$

Relations (35) and (37) imply that the tax rate needed to finance $\bar{b}$ and $\bar{t}$ equals:

$$\tau = \frac{\mathscr{M}\left[\bar{b}(1-\beta) + \bar{t}\right]}{1-\alpha} \tag{38}$$

Furthermore, (31), (34), (35) and the goods market clearing condition $Y = C$, imply that the steady-state level of labor

$$L = \left[(1-\tau)(1-\alpha)\mathscr{M}^{-1}A^{1-\sigma}\right]^{\frac{1}{\varphi+\sigma+\alpha(1-\sigma)}} \tag{39}$$

is a decreasing function of τ (given in (38)) and structural parameters.

The steady-state values of all other values can now be computed using L in (40). In particular, output equals

$$Y = A\left[(1-\tau)(1-\alpha)\mathscr{M}^{-1}A^{1-\sigma}\right]^{\frac{1-\alpha}{\varphi+\sigma+\alpha(1-\sigma)}} \tag{40}$$

(note that it is also a decreasing function of τ) and tax revenues

$$\tau \frac{W}{P} L = \tau(1-\alpha)\mathscr{M}^{-1} Y = \tau(1-\alpha)\mathscr{M}^{-1} A \Big[(1-\tau)(1-\alpha)\mathscr{M}^{-1} A^{1-\sigma}\Big]^{\frac{1-\alpha}{\varphi+\sigma+\alpha(1-\sigma)}} \tag{41}$$

Note that tax revenues are a concave function of τ.

For future reference, note in (35) that monopolistic market power distorsions can be corrected by subsidizing firm employment at the rate ε^{-1}, and financing these subsidies with lump-sum taxes. In the context of our analysis, this would imply lower (net) real steady-state transfers to the household.

9.3 Steady-state Laffer curve with textbook calibration

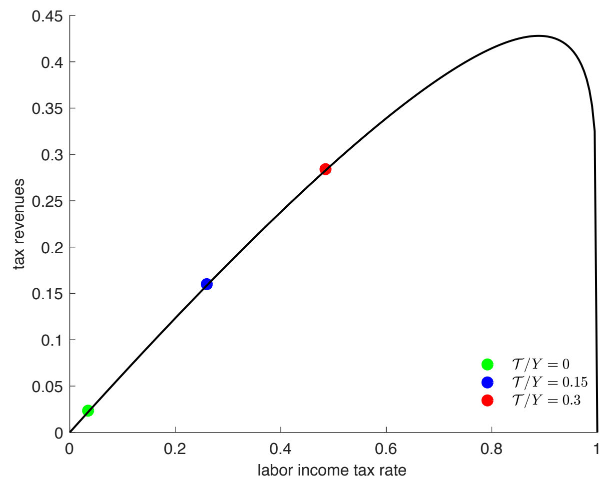

Figure 19: Steady-state Laffer curve with textbook calibration $\beta = 0.99$, $\varphi = 5$, $\theta = 0.75$, $\alpha = 0.25$, $\varepsilon = 9$, $\sigma = 1$, $\bar{b} = 2.4$ (Galí (2015))

9.4 Steady-state distorsion

The steady-state distorsion Φ is implicitly defined by

$$-\frac{U_l}{U_c} = (1-\Phi)\frac{\partial Y}{\partial L} \Rightarrow L^{\varphi} C^{\sigma} = (1-\Phi)(1-\alpha)\frac{Y}{L} \tag{42}$$

Using the good market clearing condition $C = Y$ and the production function (34), we can express the distorsion in terms of steady-state labor as

$$\Phi = 1 - \frac{A^{\sigma-1} L^{\varphi+\sigma+\alpha(1-\sigma)}}{1-\alpha}, \tag{43}$$

and further, using the expression of steady-state labor in (40), exclusively in terms of the two (steady-state) distorsionary sources- market power and distorsionary taxes:

$$\Phi = 1 - \frac{1-\tau}{\mathscr{M}} \tag{44}$$

The higher the market power in the goods market, or the higher taxes, the larger the distorsion of the steady-state allocation. When market-power distorsions are corrected (by subsidizing employment at rate ε^{-1} and financing these subsidies with lump-sum taxes)

$$\Phi = \tau \tag{45}$$

9.5 Welfare criterion large steady-state distorsions

A second order approximation around steady-state to the period-utility of the representative household,

$$U(C_t, L_t; Z_t) = \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi} \right) Z_t, \tag{46}$$

yields:

$$\begin{aligned} U_t - U \approx\; & U_c C \frac{\Delta C_t}{C} + U_l L \frac{\Delta L_t}{L} + \frac{1}{2} U_{cc} C^2 \left(\frac{\Delta C_t}{C} \right)^2 + \frac{1}{2} U_{ll} L^2 \left(\frac{\Delta L_t}{L} \right)^2 + \\ & + U_{cz} C \frac{\Delta C_t}{C} \frac{\Delta Z_t}{Z} + U_{lz} L \frac{\Delta L_t}{L} \frac{\Delta Z_t}{Z} + t.i.p. \end{aligned}$$

where $t.i.p.$ are terms independent of policy. Using further the goods market clearing condition $C_t = Y_t$, we get:

$$\begin{aligned} \frac{U_t - U}{U_c C} \approx\; & \frac{\Delta Y_t}{Y} + \frac{U_l L}{U_c C} \frac{\Delta L_t}{L} + \frac{1}{2} \frac{U_{cc} C^2}{U_c C} \left(\frac{\Delta Y_t}{Y} \right)^2 + \frac{1}{2} \frac{U_{ll} L^2}{U_c C} \left(\frac{\Delta L_t}{L} \right)^2 + \\ & + \frac{\Delta Y_t}{Y} \frac{\Delta Z_t}{Z} + \frac{U_l L}{U_c C} \frac{\Delta L_t}{L} \frac{\Delta Z_t}{Z} + t.i.p. \\ \approx\; & \frac{\Delta Y_t}{Y} \left(1 + \frac{\Delta Z_t}{Z} \right) + \frac{U_l L}{U_c C} \frac{\Delta L_t}{L} \left(1 + \frac{\Delta Z_t}{Z} \right) + \frac{1}{2} \frac{U_{cc} C^2}{U_c C} \left(\frac{\Delta Y_t}{Y} \right)^2 + \\ & + \frac{1}{2} \frac{U_{ll} L^2}{U_l L} \frac{U_l L}{U_c C} \left(\frac{\Delta L_t}{L} \right)^2 + t.i.p. \end{aligned}$$

Given the utility specification in (46), $\frac{U_{cc}C^2}{U_c C} = -\sigma$ and $\frac{U_{ll}L^2}{U_l L} = \varphi$, and hence:

$$\frac{U_t - U}{U_c C} \approx \frac{\Delta Y_t}{Y}\left(1 + \frac{\Delta Z_t}{Z}\right) + \frac{U_l L}{U_c C}\frac{\Delta L_t}{L}\left(1 + \frac{\Delta Z_t}{Z}\right) - \frac{\sigma}{2}\left(\frac{\Delta Y_t}{Y}\right)^2 + \\ + \frac{\varphi}{2}\frac{U_l L}{U_c C}\left(\frac{\Delta L_t}{L}\right)^2 + t.i.p.$$

Let Φ denote the size of the steady-state distorsion defined by $MRS = (1-\Phi)MPL$, with $MRS \equiv -\frac{U_l}{U_c}$ the steady-state marginal rate of substitution and $MPL \equiv (1-\alpha)Y/L$ the steady-state marginal product of labor[18]. The expression above can be written in terms of this distorsion as:

$$\frac{U_t - U}{U_c C} \approx \frac{\Delta Y_t}{Y}\left(1 + \frac{\Delta Z_t}{Z}\right) - (1-\Phi)(1-\alpha)\frac{\Delta L_t}{L}\left(1 + \frac{\Delta Z_t}{Z}\right) - \frac{\sigma}{2}\left(\frac{\Delta Y_t}{Y}\right)^2 \\ - \frac{\varphi(1-\Phi)(1-\alpha)}{2}\left(\frac{\Delta L_t}{L}\right)^2 + t.i.p.$$

Using the second order approximation of relative deviations in terms of log deviations $\frac{X_t - X}{X} \approx \widehat{x}_t + \frac{1}{2}\widehat{x}_t^2$, we can approximate the expression above by:

$$\begin{aligned}\frac{U_t - U}{U_c C} &\approx \left(\widehat{y}_t(1+\widehat{z}_t) + \frac{1}{2}\widehat{y}_t^2\right) - (1-\Phi)(1-\alpha)\left(\widehat{l}_t(1+\widehat{z}_t) + \frac{1}{2}\widehat{l}_t^2\right) - \frac{\sigma}{2}\widehat{y}_t^2 \\ &- \frac{\varphi}{2}(1-\Phi)(1-\alpha)\widehat{l}_t^2 + ||\mathscr{O}_t^3|| + t.i.p. \\ &\approx \widehat{y}_t(1+\widehat{z}_t) + \frac{1-\sigma}{2}\widehat{y}_t^2 - (1-\Phi)(1-\alpha)\widehat{l}_t(1+\widehat{z}_t) - \\ &- \frac{1+\varphi}{2}(1-\Phi)(1-\alpha)\widehat{l}_t^2 + ||\mathscr{O}_t^3|| + t.i.p.\end{aligned}$$

A second order approximation of the labor market clearing condition (18) yields (e.g. Galí (2015), Chapter 4): $\widehat{l}_t = \frac{\widehat{y}_t - a_t + d_t}{1-\alpha}$ with $d_t \equiv \frac{\varepsilon}{2\Theta}var_i\{p_t(i)\}$, where $\Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$, hence:

$$\frac{U_t - U}{U_c C} \approx \Phi\widehat{y}_t + \frac{1}{2}\left((1-\sigma) - \frac{(1-\Phi)(1+\varphi)}{1-\alpha}\right)\widehat{y}_t^2 - (1-\Phi)d_t + \\ + \frac{(1-\Phi)(1+\varphi)}{(1-\alpha)}a_t\widehat{y}_t + \Phi\widehat{y}_t\widehat{z}_t + t.i.p. + ||\mathscr{O}_t^3||$$

$$\frac{U_t - U}{U_c C} \approx \Phi\widehat{y}_t - \frac{1}{2}u_{yy}\widehat{y}_t^2 - \frac{1}{2}u_p var_i\{p_t(i)\} + \xi_t u_\xi \widehat{y}_t + t.i.p. + ||\mathscr{O}_t^3|| \tag{47}$$

[18] $\Phi = 1 - \frac{1-\tau}{\mathscr{M}}$

with:

$$u_{yy} \equiv \frac{(1-\Phi)(1+\varphi)}{1-\alpha} - (1-\sigma)$$
$$u_p \equiv (1-\Phi)\frac{\varepsilon}{\Theta}$$
$$\xi_t u_\xi \equiv \Phi \widehat{z}_t + \frac{(1-\Phi)(1+\varphi)}{(1-\alpha)} a_t$$

Accordingly, a second order approximation to households' welfare losses expressed as a fraction of steady-state consumption equals:

$$\begin{aligned}\mathscr{L} &= -E_0 \sum_{t=0}^{\infty} \beta^t \Big(\frac{U_t - U}{Uc\, C} \Big) \\ &= E_0 \sum_{t=0}^{\infty} \beta^t \Big(-\Phi \widehat{y}_t + \frac{1}{2} u_{yy} \widehat{y}_t^2 + \frac{1}{2} u_p var_i\{p_t(i)\} - \xi_t u_\xi \widehat{y}_t \Big) + t.i.p. + ||\mathscr{O}_t^3||\end{aligned}$$

which using $\sum_{t=0}^{\infty} \beta^t var_i\{p_t(i)\} = \frac{\theta}{(1-\beta\theta)(1-\theta)} \sum_{t=0}^{\infty} \beta^t \pi_t^2$ (Woodford (2003), Chapter 6) and the definition $u_\pi \equiv u_p \frac{\theta}{(1-\beta\theta)(1-\theta)} = (1-\Phi)\frac{\epsilon}{\lambda}$, can be written as:

$$\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Big(-\Phi \widehat{y}_t + \frac{1}{2} u_{yy} \widehat{y}_t^2 + \frac{1}{2} u_\pi \pi_t^2 - \xi_t u_\xi \widehat{y}_t \Big) + t.i.p. + ||\mathscr{O}_t^3|| \tag{48}$$

Next, we follow the approach in Benigno and Woodford (2003) to eliminate the linear term $\Phi \widehat{y}_t$. We use for this purpose a second order approximation of the aggregate supply relation. The aggregate-supply relation can be written exactly as:

$$-log\Big(\frac{1-\theta \Pi_t^{e-1}}{1-\theta} \Big) = \frac{e-1}{1+e\omega} \Big(logK_t - logF_t \Big) \quad \text{where:} \tag{49}$$

$$K_t \equiv \sum_{k=0}^{\infty} (\theta\beta)^k E_t \Big\{ \frac{1}{1-\tau_{t+k}} \frac{\mathscr{M}}{1-\alpha} \Big(\frac{Y_{t+k}}{A_{t+k}} \Big)^{\frac{1+\varphi}{1-\alpha}} \Big(\frac{P_{t+k}}{P_t} \Big)^{\frac{e(1+\varphi)}{1-\alpha}} \Big\}$$
$$F_t \equiv \sum_{k=0}^{\infty} (\theta\beta)^k E_t \Big\{ U_{c,t+k} \Big(\frac{P_{t+k}}{P_t} \Big)^{e-1} Y_{t+k} \Big\}$$
$$\omega \equiv \frac{1+\varphi}{1-\alpha} - 1$$

A second order Taylor series for the left hand-side of (49) takes the form:

$$-log\Big(\frac{1-\theta \Pi_t^{e-1}}{1-\theta} \Big) = \frac{\theta}{1-\theta}(e-1) \Big\{ \pi_t + \frac{1}{2} \frac{e-1}{1-\theta} \pi_t^2 + \mathscr{O}(||\xi||^3) \Big\},$$

whereas second order approximations for $log(K_t)$ and $log(F_t)$ imply:

$$\widehat{K_t} + \frac{1}{2}\widehat{K_t}^2 + \mathscr{O}(||\xi||^3) = (1-\theta\beta)E_t \sum_{T=t}^{\infty}(\theta\beta)^{T-t}\Big[\widehat{k}_{t,T} + \frac{1}{2}\widehat{k}_{t,T}^2\Big] + \mathscr{O}(||\xi||^3)$$
$$\widehat{F_t} + \frac{1}{2}\widehat{F_t}^2 + \mathscr{O}(||\xi||^3) = (1-\theta\beta)E_t \sum_{T=t}^{\infty}(\theta\beta)^{T-t}\Big[\widehat{f}_{t,T} + \frac{1}{2}\widehat{f}_{t,T}^2\Big] + \mathscr{O}(||\xi||^3)$$

where:

$$\widehat{k}_{t,T} \equiv \widehat{k}_T + e(1+\omega)\sum_{s=t+1}^{T}\pi_s, \quad k_T \equiv log\Big(\frac{1}{1-\tau_T}\mathscr{M}\frac{1}{1-\alpha}\Big(\frac{Y_T}{A_T}\Big)^{\frac{1+\varphi}{1-\alpha}}\Big)$$
$$\widehat{f}_{t,T} \equiv \widehat{f}_T + (e-1)\sum_{s=t+1}^{T}\pi_s, \quad f_T \equiv log(U_{Y,T}Y_T) = log(Z_T Y_T^{1-\sigma})$$

A second order approximation of $log(1-\tau_t)$ gives:

$$\begin{aligned} log(1-\tau_t) &= log(1-\tau) + \frac{(-1)\tau}{1-\tau}\frac{\Delta\tau_t}{\tau} + \frac{1}{2}(-1)\frac{-(\tau)^2}{(1-\tau)^2}(-1)\Big(\frac{\Delta\tau_t}{\tau}\Big)^2 + \\ &+ \mathscr{O}(||\xi||^3) \\ &= log(1-\tau) - \frac{\tau}{1-\tau}\widehat{\tau}_t - \frac{1}{2}\frac{\tau}{(1-\tau)^2}(\widehat{\tau}_t)^2 + \mathscr{O}(||\xi||^3), \end{aligned}$$

Hence,

$$\widehat{k}_T = \frac{1+\varphi}{1-\alpha}\Big(\widehat{y}_T - a_T\Big) + \frac{\tau}{1-\tau}\widehat{\tau}_t + \frac{1}{2}\frac{\tau}{(1-\tau)^2}(\widehat{\tau}_t)^2 + \mathscr{O}(||\xi||^3) \tag{50}$$
$$\widehat{f}_T = z_T + (1-\sigma)\widehat{y}_T \tag{51}$$

As shown by Benigno and Woodford (2004), the second order approximations of the right and left hand sides of the aggregate supply relation (49) yield the following relation:

$$\begin{aligned} \pi_t + \frac{1}{2}\frac{e-1}{1-\theta}\pi_t^2 + \frac{1}{2}(1-\theta\beta)\pi_t\mathscr{Z}_t = &\frac{1-\theta}{\theta}\frac{1-\theta\beta}{1+\omega e}\Big[(\widehat{k}_t - \widehat{f}_t) + \frac{1}{2}(\widehat{k}_t^2 - \widehat{f}_t^2)\Big] + \\ &+ \beta E_t\pi_{t+1} + \beta E_t\Big[\frac{1}{2}\frac{e-1}{1-\theta}\pi_{t+1}^2\Big] + \beta\frac{1}{2}(1-\theta\beta)E_t\pi_{t+1}\mathscr{Z}_{t+1} \\ &+ \beta\frac{1}{2}e(1+\omega)E_t\pi_{t+1}^2 + \mathscr{O}(||\xi||^3) \end{aligned} \tag{52}$$

$$\text{where} \quad \mathscr{Z}_t \equiv E_t\sum_{T=t}^{\infty}(\theta\beta)^{T-t}\Big[\widehat{k}_{t,T} + \widehat{f}_{t,T}\Big]$$

We define $V_t \equiv \pi_t + \frac{1}{2}\Big(\frac{e-1}{1-\theta} + e(1+\omega)\Big)\pi_t^2 + \frac{1}{2}(1-\theta\beta)\pi_t \mathscr{Z}_t$ and use relations (50) and (51) to compute:

$$\widehat{k}_t - \widehat{f}_t = \Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)\widehat{y}_t + \frac{\tau}{1-\tau}\widehat{\tau}_t + \frac{1}{2}\frac{\tau}{(1-\tau)^2}(\widehat{\tau}_t)^2 + t.i.p. + \mathscr{O}(||\xi||^3)$$

$$\frac{1}{2}\Big(\widehat{k}_t^2 - \widehat{f}_t^2\Big) = \frac{1}{2}\Big[\Big(\frac{1+\varphi}{1-\alpha}\Big)^2 - (1-\sigma)^2\Big]\widehat{y}_t^2 + \frac{1}{2}\Big(\frac{\tau}{1-\tau}\Big)^2(\widehat{\tau}_t)^2 + \frac{(1+\varphi)}{1-\alpha}\frac{\tau}{1-\tau}\widehat{y}_t\widehat{\tau}_t$$
$$- \Big(\frac{1+\varphi}{1-\alpha}\Big)^2\widehat{y}_t a_t - (1-\sigma)\widehat{y}_t z_t - \frac{1+\varphi}{1-\alpha}\frac{\tau}{1-\tau}\widehat{\tau}_t a_t + t.i.p. + \mathscr{O}(||\xi||^3)$$

in order to write (52) as:

$$V_t = \kappa\Big\{c_x' x_t + \frac{1}{2}x_t' C_x x_t + x_t' C_\xi \xi_t + \frac{1}{2}c_\pi \pi_t^2\Big\} + \beta E_t V_{t+1} + \mathscr{O}(||\xi||^3) + t.i.p.$$

$$\text{where:}\quad \kappa \equiv \nu_k\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big),\quad \nu_k \equiv \frac{1-\theta}{\theta}\frac{1-\theta\beta}{1+\omega e}$$

$$c_x' \equiv \Big[\frac{\tau}{1-\tau}\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)^{-1} \quad 1\Big]$$

$$x_t \equiv [\widehat{\tau}_t \;\; \widehat{y}_t]'$$

$$\xi_t \equiv [z_t \;\; a_t]'$$

$$C_x \equiv \Big[\frac{\tau(1+\tau)}{(1-\tau)^2}\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)^{-1} \quad \frac{(1+\varphi)}{1-\alpha}\frac{\tau}{1-\tau}\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)^{-1};$$
$$\frac{(1+\varphi)}{1-\alpha}\frac{\tau}{1-\tau}\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)^{-1} \quad \Big[\Big(\frac{1+\varphi}{1-\alpha}\Big)^2 - (1-\sigma)^2\Big]\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)^{-1}\Big]$$

$$C_\xi \equiv \Big[0, \quad -\frac{\tau}{1-\tau}\frac{1+\varphi}{1-\alpha}\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)^{-1};$$
$$-(1-\sigma)\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)^{-1} \quad -\Big(\frac{1+\varphi}{1-\alpha}\Big)^2\Big(\sigma + \frac{\alpha+\varphi}{1-\alpha}\Big)^{-1}\Big]$$

$$c_\pi \equiv e(1+\omega)\kappa^{-1}$$

We can integrate the equation above forward from time t to obtain:

$$\kappa^{-1}V_t = E_t\sum_{k=0}^{\infty}\beta^k\Big\{c_x' x_{t+k} + \frac{1}{2}x_{t+k}' C_x x_{t+k} + x_{t+k}' C_\xi \xi_{t+k} + \frac{1}{2}c_\pi \pi_{t+k}^2\Big\} \tag{53}$$
$$+ \mathscr{O}(||\xi||^3) + t.i.p. \tag{54}$$

Next, we determine a second order approximation to the intertemporal government solvency condition. The flow budget constraint of the government (10) implies:

$$\Pi_t^{-1}\frac{B_{t-1}}{P_{t-1}} = Q_t\frac{B_t}{P_t} + s_t, \quad s_t \equiv \tau_t\frac{W_t}{P_t}L_t - \mathscr{T}$$

where s_t is the government primary (real) surplus. This constraint can be iterated forward to get (after imposing the government public solvency condition and using the households' consumption/saving equation):

$$W_t = \sum_{k=0}^{\infty}\beta^k E_t\{U_{c,t+k}s_{t+k}\}, \quad W_t \equiv \Pi_t^{-1}\frac{B_{t-1}}{P_{t-1}}U_{c,t} \tag{55}$$

A second order approximation of $U_{c,t}s_t = Z_t C_t^{-\sigma}s_t$ gives:

$$\begin{aligned}
U_{c,t}s_t &= U_c s + Zs(-\sigma)C^{-\sigma-1}\Delta C_t + ZC^{-\sigma}\Delta s_t + \\
&+ \frac{1}{2}Zs(-\sigma)(-\sigma-1)C^{-\sigma-2}(\Delta C_t)^2 + s(-\sigma)C^{-\sigma-1}\Delta Z_t\Delta C_t + \\
&+ C^{-\sigma}\Delta Z_t\Delta s_t + Z(-\sigma)C^{-\sigma-1}\Delta C_t\Delta s_t + \mathscr{O}(||\xi||^3) + t.i.p. \\
&= C^{-\sigma}s\Big[1 - \sigma\widehat{y}_t + \frac{1}{2}\sigma^2\widehat{y}_t{}^2 + \frac{\Delta s_t}{s} - \sigma\widehat{y}_t\widehat{z}_t + \Big(\Delta Z_t - \sigma\frac{\Delta Y_t}{Y}\Big)\frac{\Delta s_t}{s}\Big] \\
&+ \mathscr{O}(||\xi||^3) + t.i.p.
\end{aligned}$$

Using the labor supply relation (5) and the goods market clearing condition $Y_t = C_t$ we can write the government primary surplus as:

$$s_t = \frac{1}{(\tau_t^{-1}-1)}Y_t^{\sigma}L_t^{\varphi+1} - \mathscr{T},$$

which can be approximated up to second order by:

$$\begin{aligned}
s_t =& s + \frac{\tau Y^{\sigma}L^{1+\varphi}}{(1-\tau)^2}\frac{\Delta\tau_t}{\tau} + \frac{\sigma Y^{\sigma}L^{1+\varphi}}{\tau^{-1}-1}\frac{\Delta Y_t}{Y} + \frac{(1+\varphi)Y^{\sigma}L^{\varphi+1}}{\tau^{-1}-1}\frac{\Delta L_t}{L} + \\
&+ Y^{\sigma}L^{1+\varphi}(1-\tau)^{-3}\tau^2\Big(\frac{\Delta\tau_t}{\tau}\Big)^2 + \frac{1}{2}\frac{\sigma(\sigma-1)Y^{\sigma}L^{1+\varphi}}{\tau^{-1}-1}\Big(\frac{\Delta Y_t}{Y}\Big)^2 + \\
&+ \frac{1}{2}\frac{(1+\varphi)\varphi Y^{\sigma}L^{1+\varphi}}{\tau^{-1}-1}\Big(\frac{\Delta L_t}{L}\Big)^2 + \frac{\tau\sigma Y^{\sigma}L^{1+\varphi}}{(1-\tau)^2}\frac{\Delta\tau_t}{\tau}\frac{\Delta Y_t}{Y} + \\
&+ \frac{\tau(1+\varphi)Y^{\sigma}L^{1+\varphi}}{(1-\tau)^2}\frac{\Delta\tau_t}{\tau}\frac{\Delta L_t}{L} + \frac{(1+\varphi)\sigma Y^{\sigma}L^{1+\varphi}}{\tau^{-1}-1}\frac{\Delta Y_t}{Y}\frac{\Delta L_t}{L} + \\
&+ \mathscr{O}(||\xi||^3)
\end{aligned}$$

$$\Rightarrow s^{-1}\Delta s_t = \omega_\tau \widehat{\tau}_t + \omega_{\tau\tau}\widehat{\tau}_t^2 + \omega_y \widehat{y}_t + \omega_{yy}\widehat{y}_t^2 + \omega_\pi var_i\{p_t(i)\} + \omega_{\tau y}\widehat{\tau}_t\widehat{y}_t +$$
$$+ \omega_{ya}\widehat{y}_t a_t + \omega_{\tau a}\widehat{\tau}_t a_t + \omega_a a_t + \mathscr{O}(||\xi||^3)$$

$$\text{with } \omega_\tau \equiv s^{-1}\frac{\tau Y^\sigma L^{1+\varphi}}{(1-\tau)^2} = (s/Y)^{-1}\frac{\tau}{1-\tau}\mathscr{M}^{-1}(1-\alpha)$$
$$\omega_{\tau\tau} \equiv s^{-1}\Big[\frac{1}{2}\frac{\tau Y^\sigma L^{1+\varphi}}{(1-\tau)^2} + Y^\sigma L^{1+\varphi}(1-\tau)^{-3}\tau^2\Big]$$
$$= (s/Y)^{-1}\frac{1-\alpha}{\mathscr{M}}\frac{\tau}{1-\tau}\Big(\frac{1}{2}+\frac{\tau}{1-\tau}\Big)$$
$$\omega_y \equiv s^{-1}\Big[\frac{\sigma Y^\sigma L^{1+\varphi}}{\tau^{-1}-1} + \frac{(1+\varphi)Y^\sigma L^{\varphi+1}}{(\tau^{-1}-1)(1-\alpha)}\Big]$$
$$= (s/Y)^{-1}\frac{(1-\tau)(1-\alpha)}{\mathscr{M}(\tau^{-1}-1)}\Big(\sigma + \frac{1+\varphi}{1-\alpha}\Big)$$
$$\omega_{yy} \equiv s^{-1}\Big[\frac{1}{2}\frac{\sigma Y^\sigma L^{1+\varphi}}{\tau^{-1}-1} + \frac{1}{2}\frac{(1+\varphi)^2 Y^\sigma L^{\varphi+1}}{(\tau^{-1}-1)(1-\alpha)^2} + \frac{1}{2}\frac{\sigma(\sigma-1)Y^\sigma L^{1+\varphi}}{\tau^{-1}-1} +$$
$$+ \frac{(1+\varphi)\sigma Y^\sigma L^{1+\varphi}}{(\tau^{-1}-1)(1-\alpha)}\Big]$$

$$= (s/Y)^{-1}\frac{(1-\tau)(1-\alpha)}{(\tau^{-1}-1)\mathscr{M}}\Big[\frac{1}{2}\frac{(1+\varphi)^2}{(1-\alpha)^2} + \frac{1}{2}\sigma^2 + \frac{\sigma(1+\varphi)}{1-\alpha}\Big]$$
$$\omega_\pi \equiv s^{-1}\frac{(1+\varphi)Y^\sigma L^{\varphi+1}}{(\tau^{-1}-1)(1-\alpha)}\frac{\varepsilon}{2\Theta}$$
$$= (s/Y)^{-1}\frac{(1+\varphi)(1-\tau)}{(\tau^{-1}-1)\mathscr{M}}\frac{\varepsilon}{2\Theta},\quad \Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon}$$
$$\omega_{\tau y} \equiv s^{-1}\Big[\frac{\tau\sigma Y^\sigma L^{1+\varphi}}{(1-\tau)^2} + \frac{\tau(1+\varphi)Y^\sigma L^{1+\varphi}}{(1-\tau)^2(1-\alpha)}\Big] = (s/Y)^{-1}\frac{\tau}{1-\tau}\frac{1-\alpha}{\mathscr{M}}\Big(\sigma+\frac{1+\varphi}{1-\alpha}\Big)$$
$$\omega_{ya} \equiv -s^{-1}\Big[\frac{(1+\varphi)Y^\sigma L^{\varphi+1}}{(\tau^{-1}-1)(1-\alpha)^2} + \frac{(1+\varphi)\varphi Y^\sigma L^{1+\varphi}}{(\tau^{-1}-1)(1-\alpha)^2} + \frac{(1+\varphi)\sigma Y^\sigma L^{1+\varphi}}{(\tau^{-1}-1)(1-\alpha)}\Big]$$
$$= -(s/Y)^{-1}\frac{(1+\varphi)(1-\tau)}{(\tau^{-1}-1)\mathscr{M}}\Big(\sigma+\frac{1+\varphi}{1-\alpha}\Big)$$
$$\omega_{\tau a} \equiv -s^{-1}\frac{\tau(1+\varphi)Y^\sigma L^{1+\varphi}}{(1-\tau)^2(1-\alpha)}$$
$$= -(s/Y)^{-1}\frac{\tau}{1-\tau}\frac{1+\varphi}{\mathscr{M}}$$
$$\omega_a \equiv -s^{-1}\frac{(1+\varphi)Y^\sigma L^{1+\varphi}}{(1-\alpha)(\tau^{-1}-1)} = -(s/Y)^{-1}\frac{(1+\varphi)(1-\tau)}{(\tau^{-1}-1)\mathscr{M}}$$

$$\text{So,}\quad U_{c,t}s_t = C^{-\sigma}s\Big[\omega_\tau\widehat{\tau}_t + (\omega_y - \sigma)\widehat{y}_t + \frac{1}{2}(\sigma^2 + 2\omega_{yy} - 2\sigma\omega_y)\widehat{y}_t^{\,2} +$$
$$+ \frac{1}{2}2\omega_{\tau\tau}\widehat{\tau}_t^2 + \frac{1}{2}2(\omega_{\tau y} - \sigma\omega_\tau)\widehat{\tau}_t\widehat{y}_t + (\omega_{ya} - \sigma\omega_a)\widehat{y}_t a_t +$$
$$+ (\omega_y - \sigma)\widehat{y}_t z_t + \omega_{\tau a}\widehat{\tau}_t a_t + \omega_\tau\widehat{\tau}_t z_t + \omega_\pi var_i\{p_t(i)\}\Big] +$$
$$+ \mathscr{O}(||\xi||^3) + t.i.p.$$

A second order approximation of W_t in (55) thus yields:

$$\Delta W_t = \sum_{k=0}^{\infty}\beta^k E_t\Big\{C^{-\sigma}s\Big[\omega_\tau\widehat{\tau}_{t+k} + (\omega_y - \sigma)\widehat{y}_{t+k} + (\frac{1}{2}\sigma^2 + \omega_{yy} - \sigma\omega_y)\widehat{y}_{t+k}^2 +$$
$$+ \omega_{\tau\tau}\widehat{\tau}_{t+k}^2 + (\omega_{\tau y} - \sigma\omega_\tau)\widehat{\tau}_{t+k}\widehat{y}_{t+k} + (\omega_{ya} - \sigma\omega_a)\widehat{y}_{t+k}a_{t+k} +$$
$$+ (\omega_y - \sigma)\widehat{y}_{t+k}z_{t+k} + \omega_{\tau a}\widehat{\tau}_{t+k}a_{t+k} + \omega_\tau\widehat{\tau}_{t+k}z_{t+k} + \omega_\pi var_i\{p_{t+k}(i)\}\Big]\Big\}$$
$$+ \mathscr{O}(||\xi||^3) + t.i.p.,$$

which can be further written as:

$$\Delta W_t = C^{-\sigma}s\Big[\omega_\tau\widehat{\tau}_t + (\omega_y - \sigma)\widehat{y}_t + \frac{1}{2}(\sigma^2 + 2\omega_{yy} - 2\sigma\omega_y)\widehat{y}_t^{\,2} + \frac{1}{2}2\omega_{\tau\tau}\widehat{\tau}_t^2 +$$
$$\frac{1}{2}2(\omega_{\tau y} - \sigma\omega_\tau)\widehat{\tau}_t\widehat{y}_t + (\omega_{ya} - \sigma\omega_a)\widehat{y}_t a_t + (\omega_y - \sigma)\widehat{y}_t z_t + \omega_{\tau a}\widehat{\tau}_t a_t +$$
$$+ \omega_\tau\widehat{\tau}_t z_t + \omega_\pi var_i\{p_t(i)\}\Big] + \beta E_t\Delta W_{t+1} + \mathscr{O}(||\xi||^3) + t.i.p.,$$

and, dividing by $W = \frac{sU_c}{(1-\beta)}$ and using the notation $\widetilde{W}_t \equiv \frac{W_t - W}{W}$, as:

$$\widetilde{W}_t = (1-\beta)\Big[b_x' x_t + \frac{1}{2}x_t' B_x x_t + x_t' B_\xi \xi_t + \omega_\pi var_i\{p_t(i)\}\Big]$$
$$+ \beta E_t\{\widetilde{W}_{t+1}\} + \mathscr{O}(||\xi||^3) + t.i.p.$$

$$\text{with}\quad b_x' \equiv [\omega_\tau \quad (\omega_y - \sigma)]$$
$$B_x \equiv [2\omega_{\tau\tau} \quad (\omega_{\tau y} - \sigma\omega_\tau); (\omega_{\tau y} - \sigma\omega_\tau) \quad (\sigma^2 + 2\omega_{yy} - 2\sigma\omega_y)]$$
$$B_\xi \equiv [\omega_\tau \quad \omega_{\tau a;}(\omega_y - \sigma) \quad (\omega_{ya} - \sigma\omega_a)]$$

Integrating this equation forward and using:

$$\sum_{t=0}^{\infty}\beta^t var_i\{p_t(i)\} = \frac{\theta}{(1-\beta\theta)(1-\theta)}\sum_{t=0}^{\infty}\beta^t\pi_t^2$$

from Woodford (2003), Chapter 6, we obtain:

$$(1-\beta)^{-1}\widetilde{W}_t = E_t \sum_{k=0}^{\infty} \beta^k \left[b_x' x_t + \frac{1}{2} x_t' B_x x_t + x_t' B_\xi \xi_t + \frac{1}{2} b_\pi \pi_t^2 \right] + \\ + \mathscr{O}(||\xi||^3) + t.i.p. \tag{56}$$

with $b_\pi \equiv 2\omega_\pi \frac{\theta}{(1-\beta\theta)(1-\theta)}$. We can now express the linear term in (47) in terms of quadratic terms by combining (53) with (56). We do so by finding ν_1 and ν_2 such that:

$$\nu_1 b_x' + \nu_2 c_x' \equiv [0 \quad \Phi],$$

$$\text{namely} \quad \nu_2 \equiv \Phi \left[1 - \frac{\tau(\omega_y - \sigma)(1-\alpha)}{(1-\tau)\omega_\tau[\sigma(1-\alpha)+\alpha+\varphi]} \right]^{-1}$$
$$\nu_1 \equiv -\frac{\tau(1-\alpha)}{(1-\tau)\omega_\tau[\sigma(1-\alpha)+\alpha+\varphi]} \nu^2$$

These two values allow to write:

$$E_t \sum_{k=0}^{\infty} \beta^k \Phi \widehat{y}_{t+k} = E_t \sum_{k=0}^{\infty} \beta^k [\nu_1 b_x' + \nu_2 c_x'] x_{t+k}$$
$$= -E_t \sum_{k=0}^{\infty} \beta^k \left[\frac{1}{2} x_{t+k}' D_x x_{t+k} + x_{t+k}' D_\xi \xi_{t+k} + \frac{1}{2} d_\pi \pi_{t+k}^2 \} \right]$$
$$+ \nu_1 (1-\beta)^{-1} \widetilde{W}_t + \nu_2 \kappa^{-1} V_t + \mathscr{O}(||\xi||^3) + t.i.p.$$

$$\text{with} \quad D_x \equiv \nu_1 B_x + \nu_2 C_x$$
$$D_\xi \equiv \nu_1 B_\xi + \nu_2 C_\xi$$
$$d_\pi \equiv \nu_1 b_\pi + \nu_2 c_\pi$$

Replacing this expression in (48), we get the following quadratic expression for welfare:

$$\mathscr{W}\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} x_t' Q_x x_t + x_t' Q_\xi \xi_t + \frac{1}{2} q_\pi \pi_t^2 \right) + T_0 + \mathscr{O}(||\xi||^3) + t.i.p.$$

$$
\begin{aligned}
\text{with} \quad & Q_x \equiv \left[D_x^{11} \quad D_x^{12} ; D_x^{21} \quad D_x^{22} + u_{yy} \right] \\
& Q_\xi \equiv \left[D_\xi^{11} \quad D_\xi^{12} ; D_\xi^{21} - \Phi \quad D_\xi^{22} - \frac{(1-\Phi)(1+\varphi)}{1-\alpha} \right] \\
& q_\pi \equiv d_\pi + u_\pi \\
& T_0 \equiv -\nu_1 (1-\beta)^{-1} \widetilde{W}_0 - \nu_2 \kappa^{-1} V_0
\end{aligned}
$$

which can be rephrased in terms of deviations from a target level of output as:

$$
\begin{aligned}
\mathscr{W}\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} Q_x^{22} \widetilde{y}_t^2 + \frac{1}{2} Q_x^{11} \widehat{\tau}_t^2 + Q_x^{12} \widetilde{y}_t \widehat{\tau}_t + \xi_t^\tau \widehat{\tau}_t + \frac{1}{2} q_\pi \pi_t^2 \right) + \\
+ T_0 + \mathscr{O}(||\xi||^3) + t.i.p.
\end{aligned}
$$

where $\widetilde{y}_t \equiv \widehat{y}_t - \widehat{y}_t^*$, $\widehat{y}_t^* \equiv -\frac{Q_\xi^{21}}{Q_x^{22}} z_t - \frac{Q_\xi^{22}}{Q_x^{22}} a_t$ and $\xi_t^\tau \equiv \left(Q_\xi^{11} - \frac{Q_x^{12} Q_\xi^{21}}{Q_x^{22}} \right) z_t + \left(Q_\xi^{12} - \frac{Q_x^{12} Q_\xi^{22}}{Q_x^{22}} \right) a_t$.

Following Eggertsson and Woodford (2004), we will rank policies in terms of the implied value of the discounted quadratic loss function:

$$
\mathscr{W}\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} Q_x^{22} \widetilde{y}_t^2 + \frac{1}{2} Q_x^{11} \widehat{\tau}_t^2 + Q_x^{12} \widetilde{y}_t \widehat{\tau}_t + \xi_t^\tau \widehat{\tau}_t + \frac{1}{2} q_\pi \pi_t^2 \right)
$$

Because this loss function is purely quadratic it is possible to evaluate it to second order using only a first order approximation to the equilibrium evolution of inflation and output under a given policy. As shown in Figure 20, Q_x^{11} and Q_x^{12} are lower than 10^{-4} under baseline calibration on the convergence path to the fiscal limit. Thus, variations in the terms $\frac{1}{2} Q_x^{11} \widehat{\tau}_t^2$ and $Q_x^{12} \widetilde{y}_t \widehat{\tau}_t$ are approximate zero up to a second order, and we can ignore them.The welfare criterion we'll thus use thereafter writes

$$
\mathscr{W}\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} Q_x^{22} \widetilde{y}_t^2 + x i_t^\tau \widehat{\tau}_t + \frac{1}{2} q_\pi \pi_t^2 \right)
$$

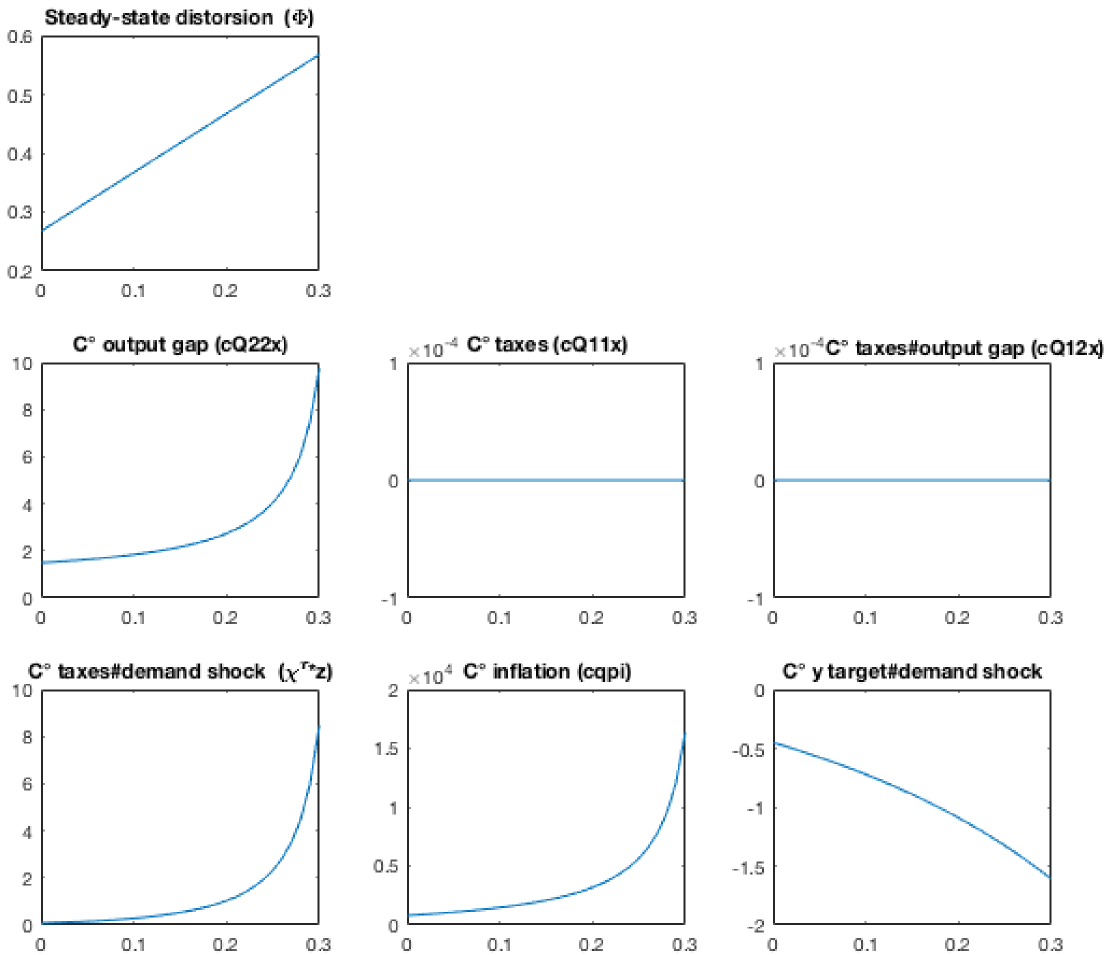

Figure 20: Policy problem and the convergence to the long-run fiscal limit

9.6 Sensitivity Analysis